Real Numbers, Indices, and Surds 9

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Focus: A set of questions and solutions for Year 9 students on Real Numbers, Indices, and Surds, tailored to the Australian Curriculum under the strand 'Number and Algebra':

1. Understanding Real Numbers:

a) Define what real numbers are and give three examples.

b) Explain the difference between rational and irrational numbers.

2. Operations with Real Numbers:

a) Simplify $3\sqrt{2} + 5\sqrt{2} - \sqrt{2}$ **.**

b) Multiply $(2 + \sqrt{3})$ **by** $(2 - \sqrt{3})$ **.**

3. Real Number Properties:

a) Demonstrate the associative property of addition with the numbers $2, 4,$ and 6.

b) Explain why the square root of a negative number is not in the set of real numbers.

4. Converting Between Forms:

a) Convert $0.666\dot{6}$ to a fraction.

b) Convert $1.25\overline{25}$ to a fraction.

c) Show that $\sqrt{16}$ is a rational number but $\sqrt{17}$ is not.

5. Real-World Application:

a) If the speed of light is approximately 3×10^8 $metres\ per\ second$, how far does light travel in $5\ seconds$?

b) A piece of string is cut into two pieces, one being $\sqrt{7}\ metres$ long. If the total length of the string is $5\ metres$, how long is the other piece?

6. Comparing Real Numbers:

a) Order these numbers from smallest to largest: $\sqrt{5},\ 2.2,\ \frac{7}{3},\ 2.25$.

b) Explain why π and 3.14 are not equal.

c) Round 3.14159 to 2 decimal places (thousandths).

d) Round 2.718 to 2 decimal place (tenths).

7. Understanding Surds

List the indices rules for multiplication, division, power of a power, negative power, and zero power.

8. Simplifying Expressions with Indices

a) Simplify $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^5$.

b) Simplify $\frac{2^5}{2^2}$ $(= 2^5 \div 2^2)$ **.**

c) Simplify $(4^2)^3$ **.**

9. Negative and Fractional Indices

a) Explain what negative indices mean. Simplify 2^{-3} **.**

b) What does a fractional exponent like $16^{1/2}$ represent? Simplify it.

c) Simplify $4^{3/2}$.

10. Understanding Surds (Indices)

a) Define what indices are.

b) Simplify $\sqrt{75}$ to its simplest surd form.

11. Operations with Surds

a) Add: $\sqrt{12} + \sqrt{3}$ **.**

b) Multiply: $\sqrt{2} \times \sqrt{32}$ **.**

c) Rationalise the denominator of $\frac{7}{\sqrt{3}}$**.**

12. Combining Indices and Surds

a) Simplify $(4\sqrt{3})^2$**.**

b) Evaluate $11^{1/3} \times 11^{2/3}$ **.** [$or \ = \sqrt[3]{11} \times \sqrt[3]{11^2} \ or \ = (\sqrt[3]{11})^1 \times (\sqrt[3]{11})^2 \ or \ = (11^{\frac{1}{3}})^1 \times (11^{\frac{1}{3}})^2$]

13. Practical Application

The length of the diagonal of a square with side length $3\sqrt{2}\,cm$ **is given by side length** $\times \sqrt{2}$ **. Calculate this diagonal length.**

14. Applying Laws of Indices in Equations

Solve for x in the equation $3^{4x} = 81$.

Solutions

1a.

See below for numbers on the real number line $\{\mathbb{R}\}$.

$$\mathbb{R} = \textit{Real Numbers} = \{\mathbb{N}^*, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}\}$$

$\mathbb{N}^*$ = Natural numbers,	$\{1, 2, 3 \ldots\}$
$\mathbb{N}$ = Natural numbers including zero,	$\{0, 1, 2, 3 \ldots\}$
$\mathbb{Z}$ = Integers,	$\{\ldots -2, -1, 0, 1, 2 \ldots\}$
$\mathbb{Q}$ = Rational numbers,	$\{\ldots \frac{-3}{5}, \frac{-1}{2}, \frac{-2}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5} \ldots\}$
$\mathbb{I}$ = Irrational numbers,	$\{\ldots -\pi, -\sqrt{3}, -\psi, -\sqrt{2}, \sqrt{2}, \psi, \sqrt{3}, \pi \ldots\}$

Real numbers are all numbers on the number line, including all rational numbers (like fractions, integers) and irrational numbers.

Examples: 3 (integer)

$\frac{1}{2}$ (rational number)

$\sqrt{2}$ (irrational number)

b.

Rational numbers can be expressed as the quotient or fraction $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

They either terminate or repeat when expressed as decimals.

Irrational numbers cannot be expressed as a simple fraction.
Their decimal expansions neither terminate nor repeat.

Examples include $\sqrt{2}, \pi,$ and e .

2a.

Since these are like terms (all have $\sqrt{2}$) :

$$\begin{aligned} &\rightarrow 3\sqrt{2} + 5\sqrt{2} - \sqrt{2} \\ &= 3\sqrt{2} + 5\sqrt{2} - 1\sqrt{2} \\ &= (3 + 5 - 1)\sqrt{2} \\ &= 7\sqrt{2}\,. \end{aligned}$$

b.

Use the difference of squares formula:

Rule for difference of two squares:

$a^2 - b^2 = (a+b)(a-b)$

$(a+b)(a-b) = a^2 - b^2$

$$\begin{aligned}(2+\sqrt{3})(2-\sqrt{3}) &= 2^2 - (\sqrt{3})^2 \\ &= 4 - (\cancel{\sqrt{}}3\cancel{)^2} \\ &= 4 - 3 \\ &= 1\,.\end{aligned}$$

3a.

Associative property of addition states $(a+b)+c = a+(b+c)$:

$$\begin{aligned}LHS &= (2+4)+6 = 6+6 \\ &= 12 \\ RHS &= 2+(4+6) = 2+10 \\ &= 12\,.\end{aligned}$$

So, $LHS \equiv RHS$

Hence, $(2+4)+6 \equiv 2+(4+6)\,.$

b.

The square root of a negative number would result in an imaginary number, like $i = \sqrt{-1}$ (i.e. $a \times a = -1$?, where a is the same number), which does not exist on the real number line where all numbers are non-negative when squared.

4a.

Let $x = 0.6666\ldots$
$= 0.\dot{6}$:

$\rightarrow 10x = 6.6666\ldots$ Choose 1 decimal place, i.e. $0.\dot{6}$ so do: $x \times 10$

Now,

$$\rightarrow 9x = 10x - x$$
$$= 6.6666... - 0.6666\ldots$$
$$= 6,$$

$$\rightarrow 9x = 6$$
$$\frac{\cancel{9}x}{\cancel{9}} = \frac{6}{9}$$
$$x = \frac{6}{9}$$
$$= \frac{6 \div 3}{9 \div 3}$$
$$= \frac{2}{3},$$
$$0.666\dot{6} = \frac{2}{3}.$$

b.

Let $x = 1.252525...$
$= 1.\overline{25}$

$\rightarrow 100x = 125.\overline{25}$ Choose 2 decimal places, so do: $x \times 100$

Now,

$$\rightarrow 99x = 100x - x$$
$$= 125.\overline{25} - 1.\overline{25}$$
$$= 124,$$

$$\rightarrow 99x = 124$$
$$\frac{\cancel{99}x}{\cancel{99}} = \frac{124}{99}$$
$$x = \frac{124}{99},$$
$$1.\overline{25} = \frac{124}{99}.$$

c.

$\sqrt{16} = 4$, which is an integer (and therefore rational) .

$\sqrt{17}$ cannot be expressed as a fraction where the numerator and denominator are integers because it's not a perfect square; its decimal goes on forever without repeating, making it irrational.

5a.

$$Speed = \frac{Distance}{Time}$$

$$\begin{aligned}
\rightarrow Speed \times Time &= \frac{Distance}{\cancel{Time}} \times \cancel{Time} \\
Speed \times Time &= Distance \\
Distance &= Speed \times Time \ :
\end{aligned}$$

$$\begin{aligned}
Distance &= 3 \times 10^8 \, m/s \times 5\, s \\
&= 3 \times 10^8 \frac{m}{\cancel{s}} \times 5\cancel{s} \\
&= (3 \times 5) \times 10^8 \, m \\
&= 15 \times 10^8 \, m \\
&= 1.5 \times 10^9 \, m\,.
\end{aligned}$$

b.

Let the length of the other piece be x :

$$\begin{aligned}
x + \sqrt{7} &= 5 \\
x + \cancel{\sqrt{7}} - \cancel{\sqrt{7}} &= 5 - \sqrt{7} \\
x &= (5 - \sqrt{7})\, metres\,.
\end{aligned}$$

Since $\sqrt{7}$ is an irrational number, the other piece will also be irrational, ensuring the sum remains 5 .

6a.

Convert to decimals for easier comparison:

$$\sqrt{5} \approx 2.236$$

$$\frac{7}{3} \approx 2.333$$

Order:

$$2.2, 2.25, \sqrt{5}, \frac{7}{3}\,.$$

b.

π is an irrational number, meaning its decimal representation goes on infinitely without repeating. 3.14 is a rational approximation of π but does not equal π due to the infinite, non-repeating nature of π.

c.

$\pi \approx 3.14159$ 9 close to 10 so round 5 up to 6 ↑

$= 3.1416$ 6 close to 10 so round 1 up to 2 ↑

$= 3.142$ 2 close to 0 so round 4 down to stay at 4 ↓

$= 3.14$. Rounded to two decimal places.

d.

$e \approx 2.718$ 8 close to 10 so round 1 up to 2 ↑

$= 2.72$. Rounded to two decimal places.

7.

Multiplication: $a^m \times a^n = a^{m+n}$

Division: $a^m \div a^n = a^{m-n}$

Power of a power: $(a^m)^n = a^{m \times n}$

Negative Power: $a^{-m} = \dfrac{1}{a^m}$

Zero Power: $a^0 = 1$

8a.

Using the multiplication law:

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^{3+5}$$ (same base, $\times$ inbetween $\rightarrow$ ADD powers)

$$= \left(\frac{1}{2}\right)^8$$

$$= \frac{1^8}{2^8}$$

$$= \frac{1}{2^8},$$

$$\rightarrow \frac{1}{256} = \frac{1}{2^2 \times 2^2 \times 2^2 \times 2^2}$$

$$= \frac{1}{4 \times 4 \times 4 \times 4}$$

$$= \frac{1}{16 \times 4 \times 4}$$

$$= \frac{1}{64 \times 2 \times 2}$$

$$= \frac{1}{128 \times 2}$$

$$= \frac{1}{256}.$$

b.

Using the division law:

$$\frac{2^5}{2^2} = 2^{5-2}$$

$$= 2^3$$

$$= 2^2 \times 2^1$$

$$= 4 \times 2$$

$$= 8.$$

c.

Using the power of a power law:

$$(4^2)^3 = 4^{2\times 3}$$

$$= 4^6$$

$$= 4^2 \times 4^2 \times 4^2$$

$$= 16 \times 16 \times 16$$

$$= 4{,}096.$$

9a.

Negative indices mean the reciprocal of the positive index.

$$2^{-3} = \frac{1}{2^3}$$
$$= \frac{1}{8}.$$

b.

A fractional exponent, $a^{1/b}$, means the b-th root of a: ($\sqrt[b]{a}$) .

$$16^{1/2} = \sqrt[2]{16}$$
$$= 4.$$

c.

$$4^{3/2} = 4^{\frac{3}{2}} = 4^{3\times\frac{1}{2}}$$
$$= (4^3)^{\frac{1}{2}}$$
$$= (32)^{\frac{1}{3}}$$
$$= (4 \times 4 \times 4)^{\frac{1}{3}}$$
$$= 4.$$

10a.

Indices (or exponents) indicate how many times a number (the base) is multiplied by itself. For example, 3^4 means 3 is multiplied by itself 4 times: $3 \times 3 \times 3 \times 3 = 81$.
They can significantly increase or decrease the magnitude of numbers.
Examples include $\sqrt{2}$ or $\sqrt[3]{5}$.

b.

$$\sqrt{75} = \sqrt{25 \times 3}$$
$$= \sqrt{25} \times \sqrt{3}$$
$$= 5 \times \sqrt{3}$$
$$= 5\sqrt{3}.$$

alternatively,

$$\sqrt{75} = 75^{\frac{1}{2}}$$
$$= (25 \times 3)^{\frac{1}{2}}$$
$$= 25^{\frac{1}{2}} \times 3^{\frac{1}{2}}$$
$$= \sqrt{25} \times \sqrt{3}$$
$$= 5 \times \sqrt{3}$$
$$= 5\sqrt{3}.$$

Remember, $x^a \times y^a = (x \times y)^a$
$or \ \ (x \times y)^a = x^a \times y^a$

$E.g.\ (5 \times 3)^2 = 5^2 \times 3^2$
$E.g.\ 7^2 \times 6^2 = (7 \times 6)^2$

11a.

In order to perform addition, we need to get both parts of the expression in terms of the same root, in this case: $\sqrt{3}$.

First, simplify $\sqrt{12}$,

$$= \sqrt{4 \times 3}$$
$$= \sqrt{4} \times \sqrt{3}$$
$$= 2 \times \sqrt{3}$$
$$= 2\sqrt{3}$$

So, $\sqrt{12} = 2\sqrt{3}$.

Then add,

$$\sqrt{12} + \sqrt{3} = 2\sqrt{3} + \sqrt{3}$$
$$= 2\sqrt{3} + 1\sqrt{3}$$
$$= 3\sqrt{3} .$$

b.

$$\sqrt{2} \times \sqrt{32} = \sqrt{2 \times 32}$$
$$= \sqrt{64}$$
$$= 8 .$$

c.

Multiply numerator and denominator by $\sqrt{3}$:

$$\frac{7}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{7 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$
$$= \frac{7 \times \sqrt{3}}{3^{\frac{1}{2}} \times 3^{\frac{1}{2}}}$$
$$= \frac{7\sqrt{3}}{3^{\frac{1}{2}+\frac{1}{2}}}$$
$$= \frac{7\sqrt{3}}{3^1}$$
$$= \frac{7\sqrt{3}}{3} .$$

Remember, $\frac{\sqrt{3}}{\sqrt{3}} = 1$, so we are multiplying by 1, which doesn't change the original expression , it only makes it look different .

Alternatively, $\sqrt{3} \times \sqrt{3} = 3$, because:

$$\sqrt{3} \times \sqrt{3} = 3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$$
$$= (3 \times 3)^{\frac{1}{2}}$$
$$= (9)^{\frac{1}{2}}$$
$$= \sqrt{9}$$
$$= 3 .$$

12a.

$$\begin{aligned}(4\sqrt{3})^2 &= (4^2)(\sqrt{3})^2\\ &= (4^2)\times 3\\ &= 16\times 3\\ &= 48\,.\end{aligned}$$

OR

$$\begin{aligned}(4\sqrt{3})^2 &= (4^2)(\sqrt{3})^2\\ &= (4^2)(3^{\frac{1}{2}})^2\\ &= (4^2)(3^{\frac{1}{2}\times 2})\\ &= 16\times 3^1\\ &= 16\times 3\\ &= 48\,.\end{aligned}$$

b.

$$\begin{aligned}11^{\frac{1}{3}}\times 11^{\frac{2}{3}} &= 11^{\frac{1}{3}+\frac{2}{3}}\\ &= 11^{\frac{1+2}{3}}\\ &= 11^{\frac{3}{3}}\\ &= 11^1\\ &= 11\,.\end{aligned}$$

13.

$$\begin{aligned}\text{Side Length}\times\sqrt{2} &= 3\sqrt{2}\times\sqrt{2}\\ &= 3\times\sqrt{2}\times\sqrt{2}\\ &= 3\times(\sqrt{2})^2\\ &= 3\times 2\\ &= 6\,cm\,.\end{aligned}$$

OR

$$\begin{aligned}\text{Side Length}\times\sqrt{2} &= 3\sqrt{2}\times\sqrt{2}\\ &= 3\times\sqrt{2}\times\sqrt{2}\\ &= 3\times(\sqrt{2})^2\\ &= 3\times(2^{\frac{1}{2}})^2\\ &= 3\times 2^{\frac{1}{2}\times 2}\\ &= 3\times 2^{\frac{2}{2}}\\ &= 3\times 2^1\\ &= 3\times 2\\ &= 6\,cm\,.\end{aligned}$$

14.

$$3^{4x} = 81$$

Recognise that:

$81 = 3^4$ so,

$$3^{4x} = 81$$

$$\rightarrow 3^{4x} = 3^4,$$

Therefore:

$$4x = 4$$

$$\frac{\cancel{4}x}{\cancel{4}} = \frac{4}{4}$$

$$x = 1\,.$$

Additional Notes for Teachers

Learning Outcomes: Students should be able to manipulate expressions involving indices and surds, understand the relationship between indices and roots, and apply these concepts in problem-solving. Students should understand the nature of real numbers, perform operations with them, recognise their properties, and apply these concepts in real-life scenarios.

Teaching Strategies: Use real-life contexts like scaling, areas, or volumes to apply indices and surds. Employ visual aids or software for demonstrating square roots and cube roots. Encourage students to check their simplifications by converting back to decimal form for verification. Use number lines, Venn diagrams, or real-life contexts to illustrate real numbers. Engage students with discussions on the nature of numbers like pi or irrational square roots.

Assessment: Evaluate through exercises requiring simplification of expressions, solving equations involving indices, and operations with surds. Test through problems that involve manipulation of real numbers, conversion between forms, and understanding their place in the number system.

Resources: Calculators for checking, but emphasise understanding over reliance on technology; worksheets with varied problems for practice. Use calculators to explore properties of real numbers, software for interactive learning about irrational numbers, or physical manipulatives for visual learning.

This question set aligns with the Australian Curriculum for Year 9, focusing on the proficiencies of understanding, fluency, problem-solving, and reasoning in number and algebra, specifically in the context of real numbers, indices, and surds.