3D Pythagoras, Spheres, Cones, and Geometric Proofs

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Focus: A set of questions and solutions for Year 9 students on Pythagoras' Theorem in 3D, tailored to the Australian Curriculum under the strand 'Measurement and Geometry':

1. Diagonal of a Rectangular Prism:

a) Find the length of the diagonal of a rectangular prism with dimensions $3\,cm$, $4\,cm$, and $5\,cm$.

b) A box has dimensions $2\,m$ by $3\,m$ by $6\,m$. What is the longest straight line you can draw inside the box?

2. Applying Pythagoras in 3D Structures:

a) A pyramid has a square base with sides of $8\,metres$ **and height of** $6\,metres$**. What is the distance from one corner of the base to the apex of the pyramid?**

b) A cuboid has sides of lengths $5\,cm,\ 12\,cm,$ **and** $13\,cm$**. Verify if the space diagonal of this cuboid forms a right triangle with the dimensions of the cuboid.**

3. Real-World Applications:

a) A ramp is built to go from the ground to the top of a platform which is $3\,metres$ high and $4\,metres$ from the base of the ramp. If the ramp is built straight and level to the ground, how long is it?

b) A cable needs to be strung from the top of a $10-metre$ tall pole to a point on the ground $15\,metres$ from the base of the pole. How long should the cable be?

4. Complex 3D Problem:

A room has dimensions $5\,m$ by $6\,m$ by $3\,m$. A spider is at one corner on the ceiling, and a fly is at the opposite corner on the floor. How far does the spider have to crawl to get to the fly if it can only crawl along the walls, floor, and ceiling, taking the minimum path?

5. Spheres

a) Find the volume of a sphere with radius $3\,cm$.

b) Find the surface area of a sphere with a diameter of $10\,cm$. Use $\pi \approx 3.14$.

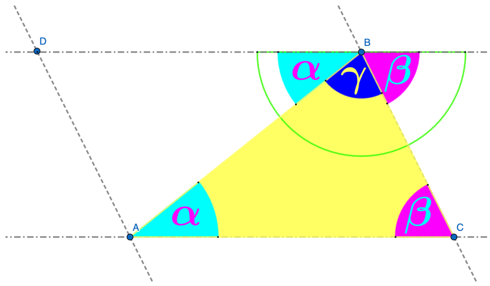
6. Cones

a) Calculate the volume of a cone with a radius of $2\,cm$ and a height of $5\,cm$. Use $\pi \approx 3.14$.

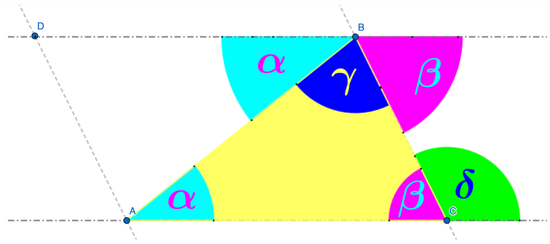
b) A cone has a base radius of $5\,cm$ and a slant height of $12\,cm$. Calculate the surface area of the cone (excluding the base). Use $\pi \approx 3.14$.

7. Basic Angle Properties:

a) Prove that the sum of angles in a triangle equal 180°.

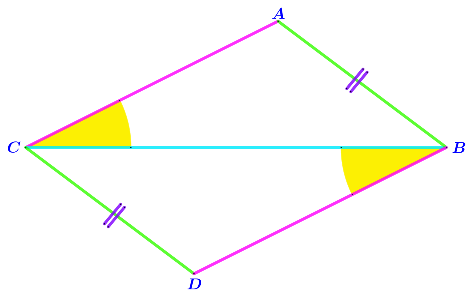


b) Prove that the exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles; i.e. show $\delta = \alpha + \gamma$.

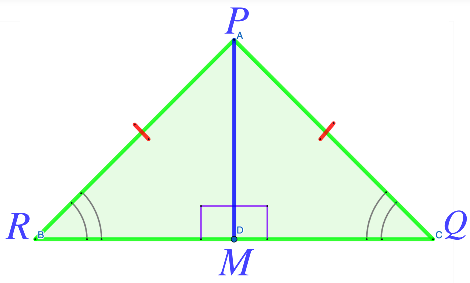


8. Congruent Triangles:

a) Prove that two triangles are congruent if they have the same side lengths (SSS).

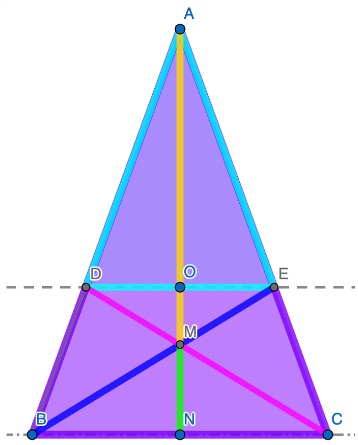


b) In $\triangle PQR$, $PQ = PR$, and M is the midpoint of QR. Prove that $\triangle PMQ \cong \triangle PMR$.

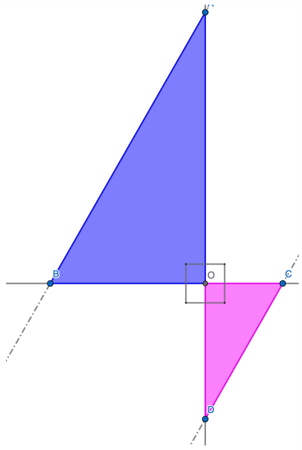


9. Similar Triangles:

a) Prove the Basic Proportionality Theorem: If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.

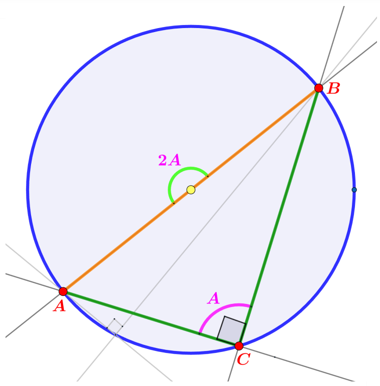


b) Prove that if two triangles are similar, their corresponding sides are proportional.



10. Circle Theorems:

Prove that the angle subtended by a semicircle is a right angle.

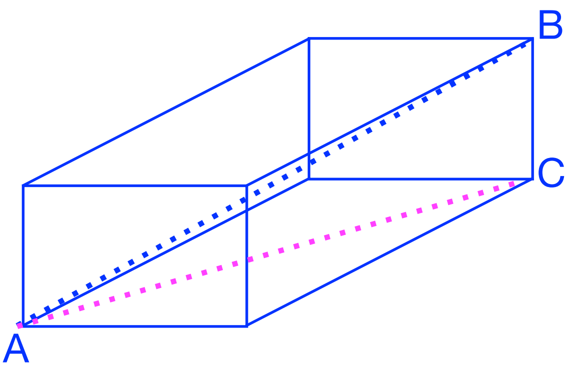


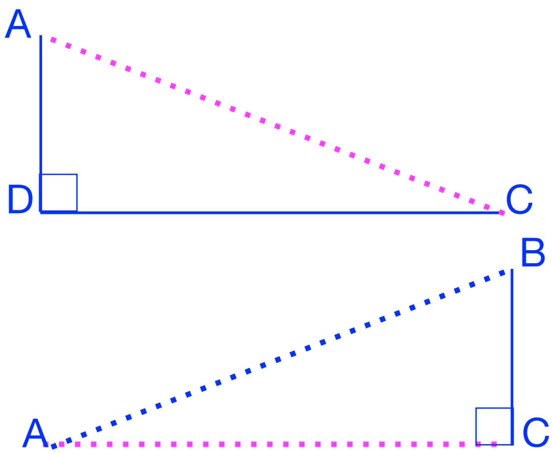
Solutions

$$d^2 = a^2 + b^2 + c^2,$$

$$\rightarrow d = \sqrt{a^2 + b^2 + c^2}, \equiv d = \sqrt{x^2 + y^2 + z^2}.$$

$$AB = \sqrt{(AD)^2 + (DC)^2 + (CB)^2}.$$





1a.

The diagonal in a 3D space can be found using the extension of the Pythagorean theorem for three dimensions.

Let's denote the diagonal as d :

$$d = \sqrt{l^2 + w^2 + h^2}$$

Here, $l = 3,\ w = 4,\ h = 5$:

$$d = \sqrt{3^2 + 4^2 + 5^2}$$

$$d = \sqrt{9 + 16 + 25}$$

$$d = \sqrt{50}$$

$$\approx 7.07\,cm.$$

b.

The longest line inside a rectangular prism is the space diagonal:

$$d = \sqrt{a^2 + b^2 + c^2}$$

$$d = \sqrt{2^2 + 3^2 + 6^2}$$

$$= \sqrt{4 + 9 + 36}$$

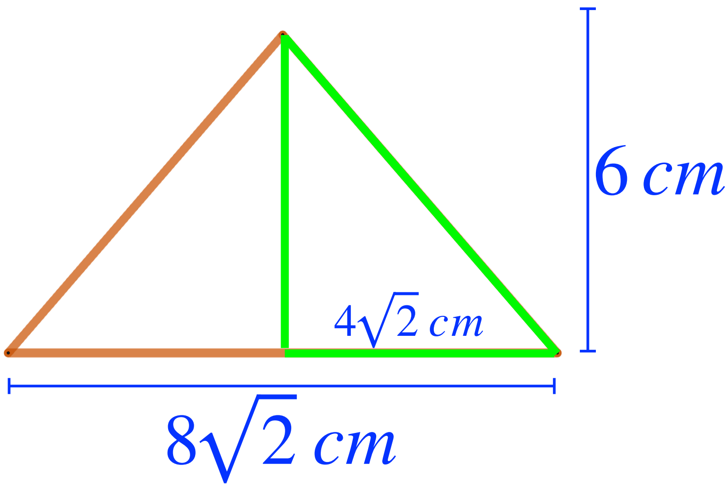
$$= \sqrt{49}$$

$$d = 7\,metres.$$

2a.

First, find the diagonal of the base:

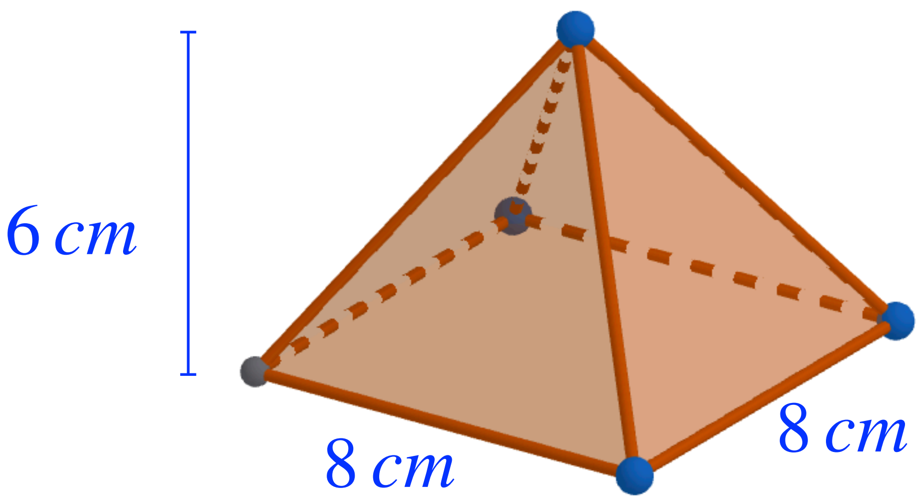
$$\text{Diagonal of square base} = \sqrt{8^2 + 8^2}$$



$$c = \sqrt{a^2 + b^2}$$
$$= \sqrt{64 + 64}$$
$$= \sqrt{128}$$
$$= \sqrt{64 \times 2}$$
$$= \sqrt{64} \times \sqrt{2}$$
$$= 8\sqrt{2}\,metres.$$

Now, use half this diagonal as one leg and the height as the other:

$$= (8\sqrt{2}) \div 2$$
$$= 4\sqrt{2}$$
$$c = \sqrt{a^2 + b^2}$$
$$\text{Distance to apex} = \sqrt{(4\sqrt{2})^2 + 6^2}$$
$$= \sqrt{4^2 \times \left(\left(\cancel{\sqrt{}}2\right)^{\cancel{2}} + 6^2\right.}$$
$$= \sqrt{16 \times 2 + 36}$$
$$= \sqrt{32 + 36}$$
$$= \sqrt{68}$$
$$\approx 8.25\,metres.$$



b.

Determine the length of the space diagonal

$$a^2 + b^2 + c^2 = d^2$$

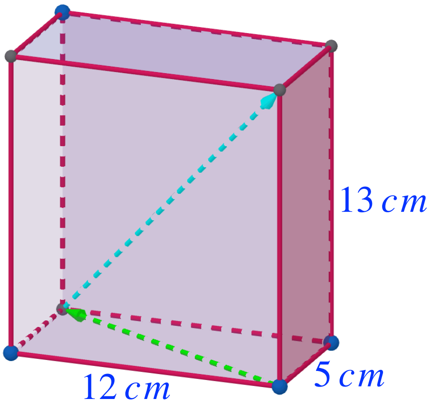
$$d^2 = 5^2 + 12^2 + 13^2$$
$$= 25 + 144 + 169$$
$$\cancel{\sqrt{d^2}} = \sqrt{338}$$
$$d = \sqrt{338}$$
$$d = 169\,cm\,.$$

So, the space diagonal is $169\,cm$.

Now, check $5 \times 12 \times 13$ will fit the Pythagoran Theorem in 3D

Check if the diagonal d satisfies:

$$d^2 = a^2 + b^2 + c^2$$
$$d = \sqrt{a^2 + b^2 + c^2}$$
$$LHS = d$$
$$RHS = \sqrt{a^2 + b^2 + c^2}$$



$$LHS = d = 169$$

$$RHS = \sqrt{a^2 + b^2 + c^2} = \sqrt{5^2 + 12^2 + 13^2}$$
$$= \sqrt{25 + 144 + 169}$$
$$= \sqrt{338}$$
$$= 169$$

$$\therefore LHS = RHS$$

So, the dimensions $5 \times 12 \times 13$ fit the Pythagorean Theorem in 3D.

3a.

The ramp forms the hypotenuse of a right triangle where:

Base $= 4\, metres$
Height $= 3\, metres$

Length of ramp :

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5\, metres\,. \end{aligned}$$

b.

Here, we're looking at a right triangle where:
One leg is the height of the pole ($10\, metres$)
The other leg is the distance from the base to the point on the ground ($15\, metres$)

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c &= \sqrt{a^2 + b^2} \\ \text{Cable length } &= \sqrt{10^2 + 15^2} \\ &= \sqrt{100 + 225} \\ &= \sqrt{325} \\ &\approx 18.03\, metres\,. \end{aligned}$$

4.

The spider must traverse the space diagonal of the room, but since it can only crawl on surfaces, it essentially makes two moves: First from the top corner to the bottom of the same vertical line, then along the floor to the fly. However, the direct diagonal is more straightforward:

$$\begin{aligned} d &= \sqrt{a^2 + b^2 + c^2} \\ d &= \sqrt{5^2 + 6^2 + 3^2} \\ &= \sqrt{25 + 36 + 9} \\ &= \sqrt{70} \\ d &\approx 8.37\, metres\,. \end{aligned}$$

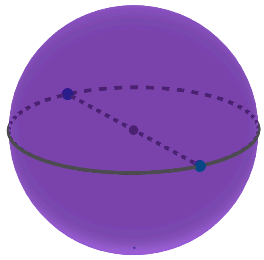
5a.

$$V_{sphere} = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times 3.14 \times 3^3$$
$$= \frac{4}{3} \times 3.14 \times 27$$
$$\approx 113\,cm^3.$$

b.

Radius, $r = \frac{10}{2}$
$= 5\,cm$,

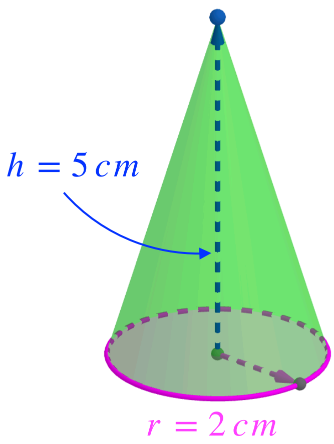
$$A_{sphere} = 4\pi r^2$$
$$= 4 \times 3.14 \times 5^2$$
$$= 4 \times 3.14 \times 25$$
$$= 314\,cm^2.$$



6a.

$$V_{cone} = \frac{1}{3}\pi r^2 h$$
$r = 2\,cm$.
$h = 5\,cm$.

$$= \frac{1}{3} \times 3.14 \times 2^2 \times 5$$
$$= \frac{1}{3} \times 3.14 \times 4 \times 5$$
$$= 20.9\dot{3}\,cm^3.$$



b.

$$SA_{cone} = \pi r l$$
$r = 5\,cm$.
$l = 12\,cm$.

The surface area of the cone (excluding the base) involves only the lateral surface area:

[Lateral Surface Area] $= \pi r l$
$= 3.14 \times 5 \times 12$
$= 188.4\,cm^2$.

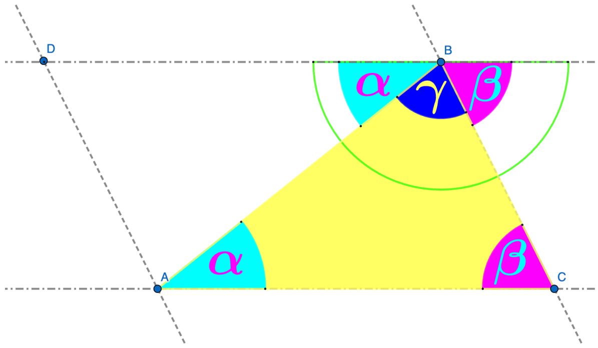
7a.

Consider a triangle ABC.

Draw a line parallel to AC through point B.

Draw a line parallel to BC through point A.

Label the intersection of these two lines as D.



Since $DB \parallel AC$, the alternate interior angles are equal:

$\alpha = \alpha$ and $\beta = \beta$.

The angles on a straight line sum to 180°:

So, $180^\circ - (\alpha + \beta) = \gamma$

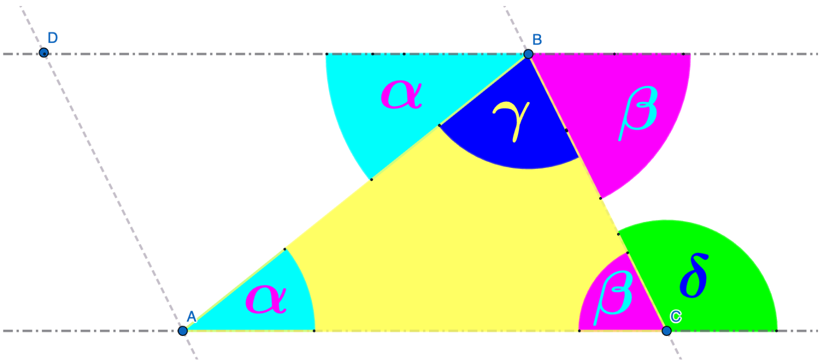
$\rightarrow 180^\circ \cancel{-(\alpha+\beta)} \cancel{+(\alpha+\beta)} = \gamma \; +(\alpha + \beta)$

$180^\circ = \gamma + \alpha + \beta$

$\therefore \alpha + \beta + \gamma = 180^\circ$.

Hence, the sum of the angles in any triangle is 180°.

b.



Let the triangle be ABC with an exterior angle, δ.

By the angle sum in a triangle:

$\alpha + \beta + \gamma = 180^\circ$.

$\delta + \beta = 180^\circ$ since they form a linear pair.

Since they both equal 180° we set the equations equal to each other :

$\alpha + \beta + \gamma = \delta + \beta$

$\rightarrow \alpha + \cancel{\beta} + \gamma = \delta + \cancel{\beta}$

$\rightarrow \alpha + \gamma = \delta$

Thus, the exterior angle (δ) equals the sum of the two non-adjacent interior angles $(\alpha$ and $\beta)$.

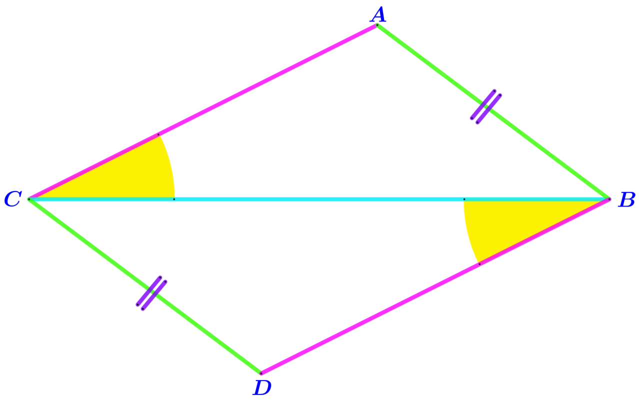
8a.

Consider triangles ABC and DCB , where $AB = CD,$ and $CB = BC$.

$\therefore CA = DB$

Since all corresponding sides are equal, the triangles must have the same shape and size, and thus they must be congruent with each other.

Hence, by the definition of congruence, $\triangle ABC \cong \triangle DCB$.
Thus proving the $(Side - Side - Side)$ rule.



b.

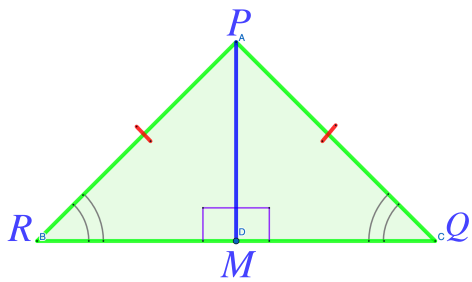
Given $PQ = PR$, $\triangle PQR$ is isosceles with M as the midpoint of QR .

PM is a common side to both $\triangle PMQ$ and $\triangle PMR$.

$PQ = PR$ (given).

$QM = MR$ (since M is the midpoint).

By Side-Side-Side (SSS) congruence criterion:
$\therefore \triangle PMQ \cong \triangle PMR$.



9a.

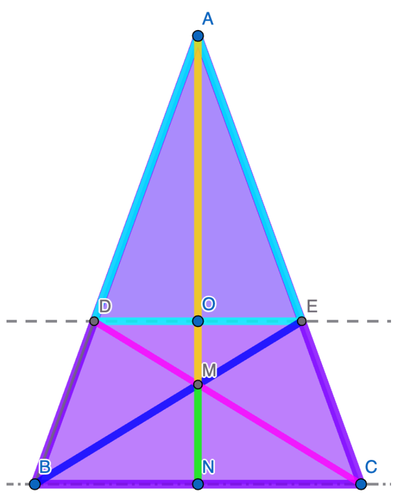
Consider $\triangle ABC$ with line DE parallel to BC intersecting AB at D and AC at E .

We need to prove $\dfrac{AD}{DB} = \dfrac{AE}{EC}$. Start by drawing a line down from point A to the midpoint of BC .

Draw a line from $D \rightarrow C$, and a line from $E \rightarrow B$.

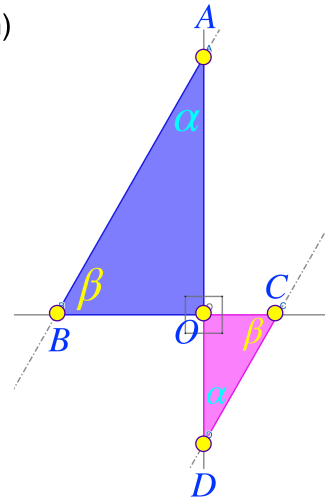
Since $DE \parallel BC$, quadrilateral $DBCE$ is a trapezium, and
$\triangle AOD \cong \triangle AOE$ and $\triangle DMB \cong EMC$

$\therefore$ the ratio of: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.



b.

Consider two similar triangles ABC and DEF (by AAA, similarity criterion)



By definition of similarity, the corresponding angles are equal, and the triangles are the same shape.

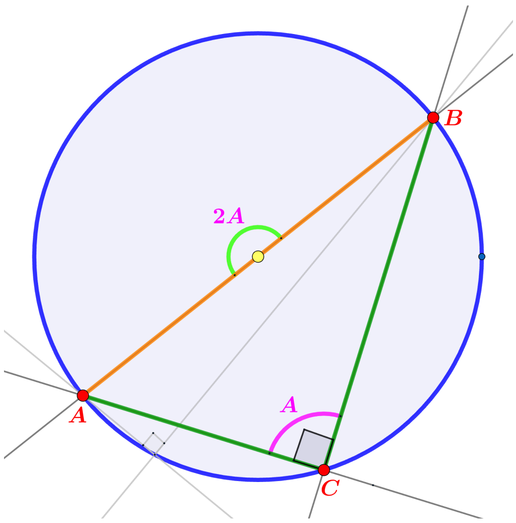
Construct a ratio of corresponding sides:

$$\frac{AB}{AO} = \frac{DC}{DO}, \frac{OB}{OA} = \frac{OC}{OD}, \frac{OB}{BA} = \frac{OC}{CD}$$

These ratios being constant for each pair of corresponding sides prove the proportionality of sides in similar triangles.

10.

Consider a semicircle with diameter AB, and an inscribed angle $\angle ACB$ (pink line). Draw the line from C to A and C to B (dark green lines).



Since AB (orange line) is the diameter, $\angle ACB$ (magenta angle) subtends the arc at AB (orange line), i.e. $\angle AOB$ (bright green angle), with O being the circle centre . The angle at the centre (bright green line) subtended by this arc is 180° (since it's a semicircle).

The angle at the circumference is half the angle at the centre for the same arc, i.e a right-angle (reverse of double angle theorem, [see diagram below]), so:

$$\begin{aligned}\angle ACB &= \frac{180^\circ}{2} \\ &= 90^\circ.\end{aligned}$$

Additional Notes for Teachers:

Learning Outcomes:

Students should be able to apply Pythagoras' theorem to solve problems in three dimensions, understanding how to find lengths of diagonals in rectangular prisms and other 3D shapes. Students should understand and be able to construct geometric proofs, demonstrating logical reasoning and understanding of geometric properties.

Teaching Strategies:

Use models or virtual reality to visualise 3D space. Encourage students to construct their own 3D problems or scenarios. Use diagrams and step-by-step reasoning to illustrate proofs. Encourage students to use geometric tools for constructing their own proofs.

Assessment:

Evaluate through problems where students must determine distances or verify geometric relationships in 3D space. Test through exercises that require proving geometric properties, using congruence and similarity criteria, and applying circle theorems.

Resources:

Geometry software for dynamic visualisation, physical models like cubes or prisms for hands-on learning, or real-life scenarios like architectural designs. Geometry software for dynamic illustrations, physical drawing tools like compasses and rulers, and structured worksheets for guided proof practice.

This set of questions aligns with the Australian Curriculum for Year 9, focusing on applying Pythagorean principles in three-dimensional contexts.