Trigonometry, and Transformations 9

Free and always will be!

Focus: A set of questions and solutions for Year 9 students on Trigonometry, and Transformations tailored to the Australian Curriculum under the strand 'Measurement and Geometry':

1. Basic Trigonometric Ratios:

a) If θ is an angle in a right-angled triangle with opposite side length $3\,cm$ and adjacent side length $4\,cm$, find $\sin\theta$, $\cos\theta$, and $\tan\theta$.

b) Given $\sin\theta = \frac{5}{13}$ **and** θ **is an acute angle, find** $\cos\theta$ **and** $\tan\theta$ **.**

2. Using Trigonometry in Right Triangles:

a) A ladder leans against a wall at an angle of 60° **. If the base of the ladder is** $2\,metres$ **from the wall, how high does the ladder reach?**

b) From the top of a $50-metre$ high building, the angle of depression to a car on the ground is 30°. How far is the car from the base of the building?

3. Angles of Elevation and Depression:

a) An airplane is flying at an altitude of $8\,km$. If the angle of elevation from a point on the ground to the airplane is 45°, how far is the airplane horizontally from that point?

b) A hiker can see the top of a mountain at an angle of elevation of 25° from his current position. If the hiker is $3\,km$ from the base of the mountain, how tall is the mountain?

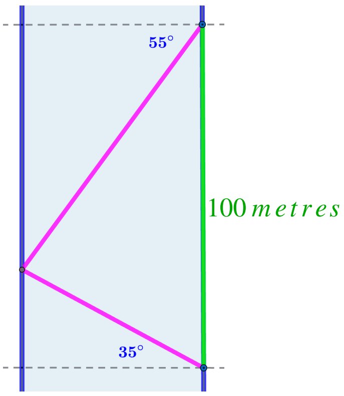
4. Non-Right Triangles - Sine and Cosine Rules:

a) In triangle ABC, $AC = 6\,cm$, $BC = 5\,cm$, and $\angle ACB = 30^\circ$. Find the length of AB using the cosine rule.

b) Find $\angle B$ in the same triangle ABC using the sine rule.

5. Real-World Applications: *(Tricky)*

A surveyor needs to find the width of a river. He stands on one bank, looks directly across the river and measures the angle up to a point across the river to be 35°, then walks $100\,metres$ along the bank and measures the angle down to the same point to be 55°. How wide is the river?



6. Area Rule

Area Rule: $Area = \frac{1}{2}ab\sin(C)$

a) Find the area of a triangle with side lengths $a = 10$, $b = 12$, and the angle between a and b is 20°

b) Given the area of a triangle is $15\,cm^2$, and has side lengths $a = 12\,cm,\ b = ?$, with the angle between a and b, $\angle ab = 30^\circ$, find the missing side length b.

7. Transformations

a) A point $P(3, 4)$ is reflected over the line $y = x$. What are the coordinates of the image point P'?

b) A triangle with vertices $A(1, 2)$, $B(3, 2)$, and $C(2, 4)$ is translated $5\ units$ to the right and $2\ units$ down. Write the coordinates of the vertices of the translated triangle $A'B'C'$.

c) A square with vertices $S(-2,\ 1)$, $T(1,\ 1)$, $U(1,\ -2)$, and $V(-2,\ -2)$ is rotated 90° anticlockwise about the origin. Determine the coordinates of the image square $S'T'U'V'$, and explain how the rotation affects the coordinates.

d) A designer is creating a tessellation pattern using a regular hexagon. The hexagon is first translated by the vector $(4, 0)$, then reflected over the line $y = 0$ (the $x - axis$). If one vertex of the original hexagon is at $(1, 2)$, find the coordinates of this vertex after both transformations, and explain how these transformations contribute to the tessellation.

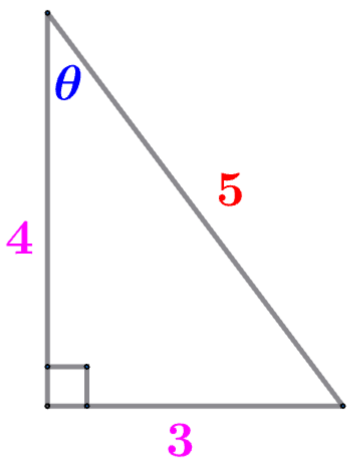
Solutions

1a.

Firstly, using the Pythagorean theorem, find the $hypotenuse$:

$$c = \sqrt{a^2 + b^2}$$

$$\begin{aligned} hypotenuse &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5\,cm\,. \end{aligned}$$



SOH CAH TOA

Now calculate:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{3}{5}\ \left(= 0.6\ \right).$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{4}{5}\ \left(= 0.8\ \right).$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

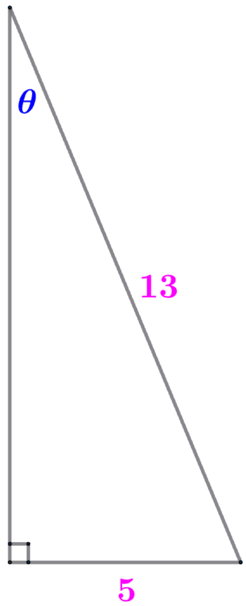
$$= \frac{3}{4}\ \left(= 0.75\ \right).$$

b.

Since, $\sin\theta = \dfrac{Opposite}{Hypotenuse} = \dfrac{5}{13}$, *SOH CAH TOA*

the opposite side is 5 and the hypotenuse is 13.

Use the Pythagorean theorem to find the adjacent side:



Let a be the adjacent side, then:

$$a^2 + b^2 = c^2$$
$$a^2 + 5^2 = 13^2$$
$$a^2 + 25 = 169$$
$$a^2 \cancel{+25} \cancel{-25} = 169 - 25$$
$$a^2 = 144$$
$$\cancel{\sqrt{}}\,a^{\cancel{2}} = \sqrt{144}$$
$$a = 12\,,$$

Now calculate:

$$\cos\theta = \frac{12}{13}$$
$$\tan\theta = \frac{5}{12}\,.$$

2a.

Find height :

$$\tan\theta = \frac{Opposite}{Adjacent}$$

$$\tan 60^\circ = \frac{\text{height}}{2}$$

$$\sqrt{3} = \frac{h}{2}$$

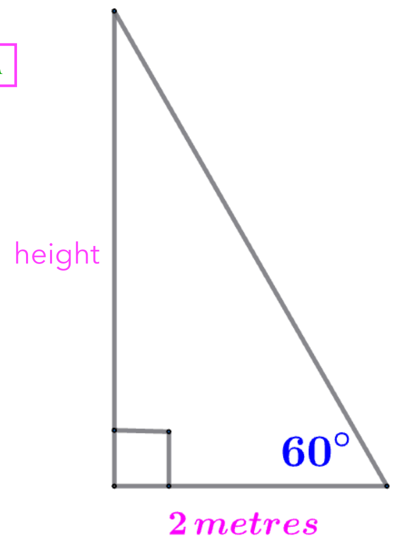
$$\sqrt{3} \times 2 = \frac{h}{\cancel{2}} \times 2$$

$$2\sqrt{3} = h$$

$$h = 2\sqrt{3}$$

$$\approx 3.46\, meters.$$

$SOH\ CAH\ \boxed{TOA}$

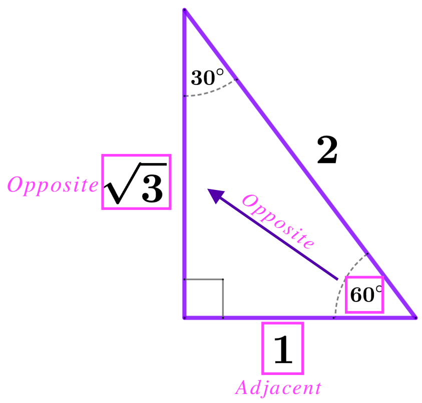


Using,

$$\tan\theta = \frac{Opposite}{Adjacent}$$

$$\tan 60^\circ = \frac{\sqrt{3}}{1}$$

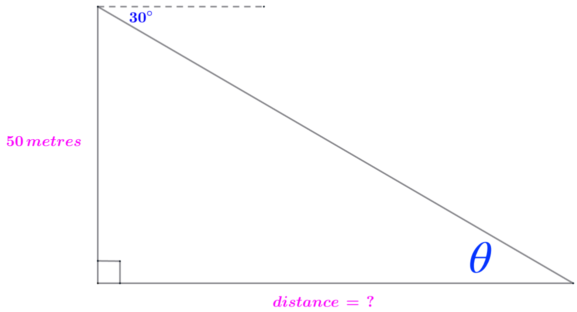
$$\tan 60^\circ = \sqrt{3}\ (\approx 1.732)$$



b.

The angle of depression is equal to the angle of elevation from the car to the top of the building, so we use:

$$\tan 30^\circ = \frac{1}{\sqrt{3}} :$$



$$\tan\theta = \frac{Opposite}{Adjacent}$$

SOH CAH $\boxed{\text{TOA}}$

$$\tan 30^\circ = \frac{50}{d}$$

$$\frac{1}{\sqrt{3}} = \frac{50}{d}$$

$$\frac{1}{\cancel{\sqrt{3}}} \times \cancel{\sqrt{3}} = \frac{50}{d} \times \sqrt{3}$$

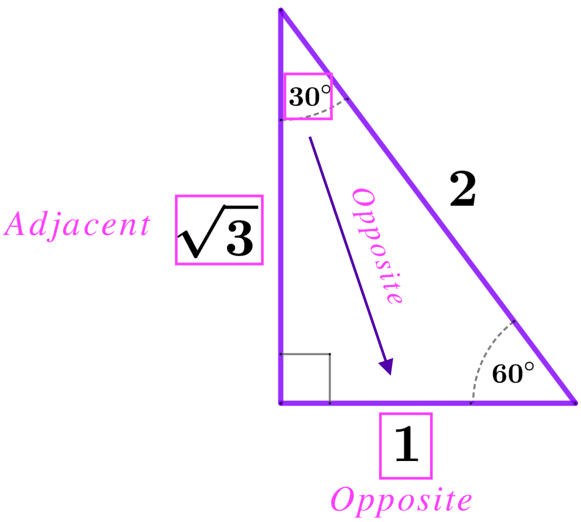
$$1 = \frac{50}{d} \times \sqrt{3}$$

$$1 = \frac{50\sqrt{3}}{d}$$

$$1 \times d = \frac{50\sqrt{3}}{\cancel{d}} \times \cancel{d}$$

$$d = 50\sqrt{3}$$

$$\approx 86.6\,meters.$$



$$\tan(\theta) = \frac{Opp}{Adj.}$$

$$\tan(30^\circ) = \frac{Opp.}{Adj.}$$

$$\frac{Opp.}{Adj.} = \tan(30^\circ)$$

$$\approx 1.732 \approx \frac{1}{\sqrt{3}} = \tan(30^\circ)$$

3a.

Here,

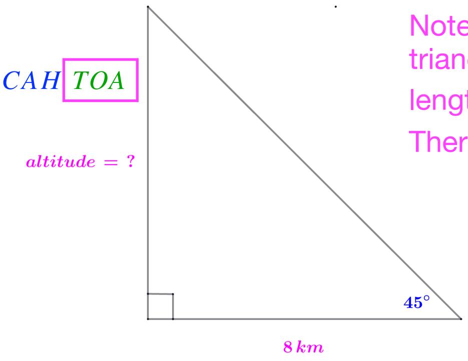
$\tan 45° = 1$, so :

$$\tan\theta = \frac{Opposite}{Adjacent} \quad SOH\ CAH\ \boxed{TOA}$$

$$1 = \frac{8}{d}$$

$$1 \times d = \frac{8}{\cancel{d}} \times \cancel{d}$$

$$d = 8\,km\,.$$



Note: As the angle is $45°$, this means the triangle formed is a square with side lengths $8\,km$, cut through the diagonal. Therefore, the altitude must be $8\,km$.

b.

Using : $\tan 25° \approx 0.466$:

$$\tan\theta = \frac{Opposite}{Adjacent} \quad SOH\ CAH\ \boxed{TOA}$$

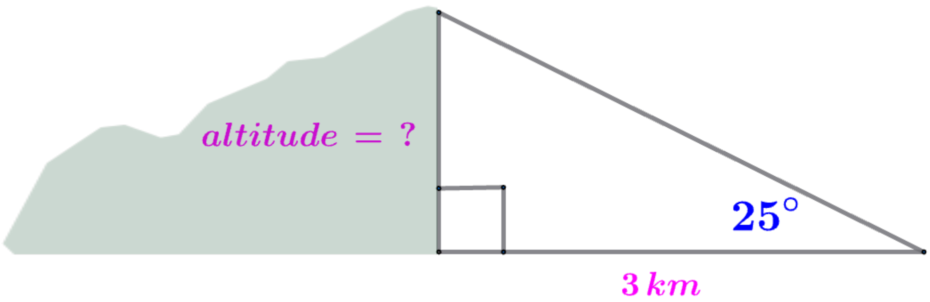
$$\tan 25° = \frac{h}{3}$$

$$0.466 \approx \frac{h}{3}$$

$$0.466 \times 3 \approx \frac{h}{\cancel{3}} \times 3$$

$$1.398 \approx h$$

$h \approx 1.398\,km$, or approximately $1.4\,km$.



4a.

Cosine rule:

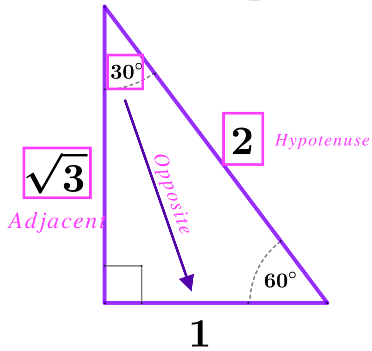
SOH CAH TOA

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c = ?,\ a = 5,\ b = 6,\ C = 30^\circ$$

$$\cos(\theta) = \frac{Adj.}{Hyp.}$$

$$\cos C = \cos 30^\circ = \frac{\sqrt{3}}{2} \longleftarrow \cos(30^\circ) = \frac{\sqrt{3}}{2}$$



$$c^2 = 5^2 + 6^2 - 2\times 5\times 6\times\frac{\sqrt{3}}{2}$$

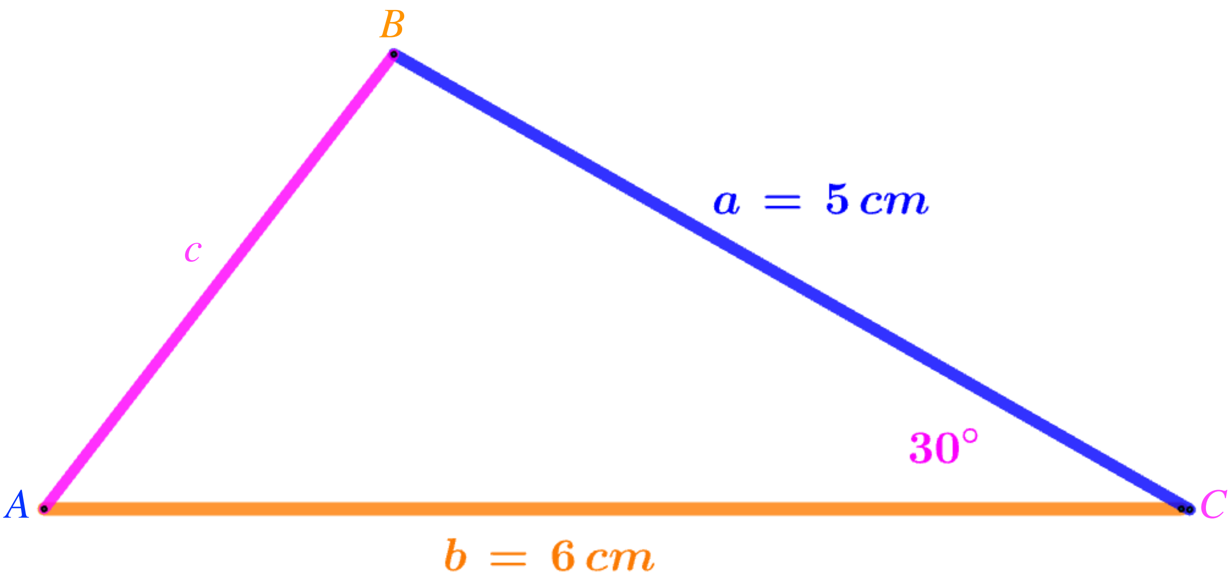
$$c^2 = 25 + 36 - 30\sqrt{3}$$

$$c^2 = 61 - 30\sqrt{3}$$

$$\cancel{\sqrt{}}\,c^{\cancel{2}} = \sqrt{61 - 30\sqrt{3}}$$

$$c = \sqrt{61 - 30\sqrt{3}}$$

$$\approx 3\,cm.$$



b.

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

We know $a = 5,\ b = 6,\ c \approx 3,\ \angle C = 30^\circ$, and $\sin 30^\circ = 0.5$, so:

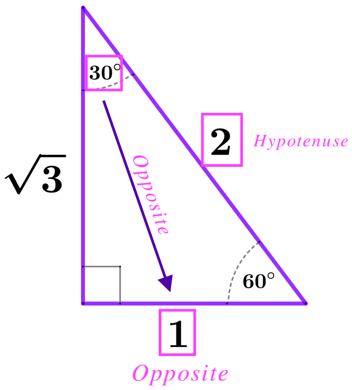
$$\begin{aligned}
\frac{b}{\sin B} &= \frac{c}{\sin C} \\
\frac{6}{\sin B} &\approx \frac{3}{\sin 30^\circ} \\
\frac{6}{\sin B} &\approx \frac{3}{0.5} \\
\frac{6}{\sin B} &\approx 6 \\
6 &\approx \frac{6}{\sin B} \\
6 \times \sin B &\approx \frac{6}{\cancel{\sin B}} \times \cancel{\sin B} \\
6 \sin B &\approx 6 \\
\frac{\cancel{6} \sin B}{\cancel{6}} &\approx \frac{6}{6} \\
\sin B &\approx 1 \\
\cancel{\sin^{-1}}(\cancel{\sin} B) &\approx \sin^{-1}(1) \\
\angle B &\approx \sin^{-1}(1) \\
&\approx 90^\circ .
\end{aligned}$$

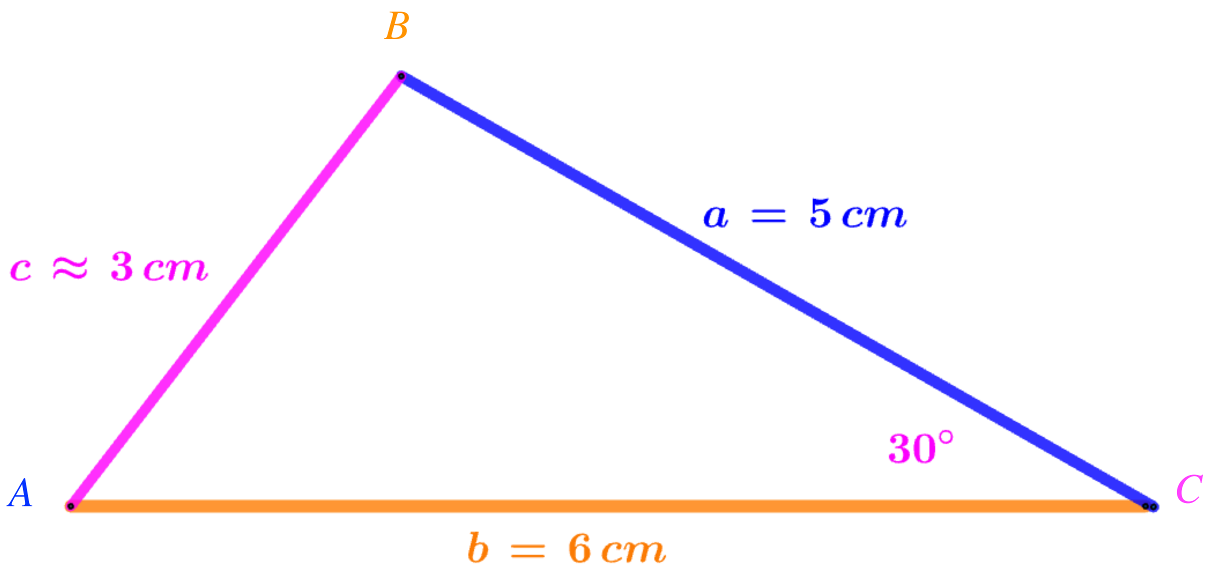
SOH CAH TOA

$$\sin(\theta) = \frac{Opp.}{Hyp.}$$

$$\sin(30^\circ) = \frac{1}{2}$$

$$\sin(30^\circ) = 0.5$$





5.

As we know he measured from *directly* across the river bed, this makes the angle between where he measures from up to the side of the river to be exactly 90° . This means, we can find the angles inside the triangle by taking each one away from 90°.

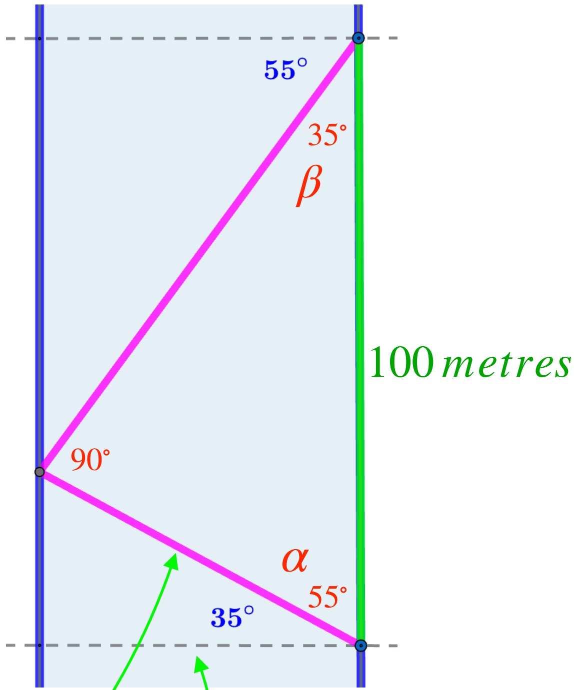
This gives:

$$\alpha = 90 - 35$$
$$\alpha = 55^\circ .$$

$$\beta = 90 - 5$$
$$\beta = 35^\circ .$$

Which means the third angle is:

$$\begin{aligned} &180^\circ - (55^\circ + 35^\circ) \\ &= 180^\circ - 90^\circ \\ &= 90^\circ . \end{aligned}$$



Let d be the distance across the river from the starting point :

SOH CAH TOA

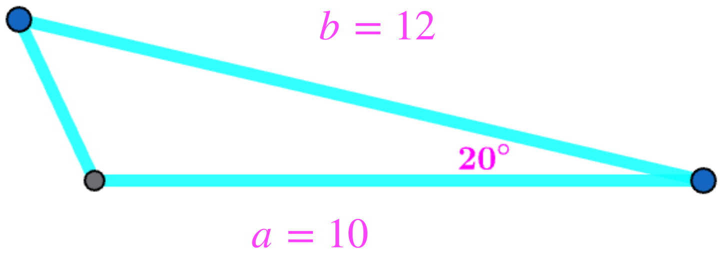
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{d}{\sin 35} = \frac{100}{\sin 90}$$
$$\frac{d}{\sin 35} = \frac{100}{1}$$
$$\frac{d}{\cancel{\sin 35}} \times \cancel{\sin 35} = 100 \times \sin 35$$
$$d = 100 \times \sin 35^\circ$$
$$\approx 57.36\, metres .$$

We now have enough information to find the width of the river w :

$$\cos(\theta) = \frac{Adjacent}{Hypotenuse}$$
$$\cos(35) = \frac{w}{57.36}$$
$$\cos(35) \times 57.36 = \frac{w}{\cancel{57.36}} \times \cancel{57.36}$$
$$\cos(35) \times 57.36 = w$$
$$w = \cos(35) \times 57.36$$
$$= 57.36 \times \cos(35)$$
$$w \approx 46.99\, m .$$

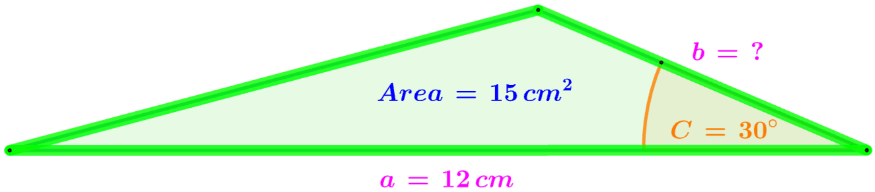
6a.

$$a = 10$$
$$b = 12$$
$$C = 20^\circ$$
$$Area = ?$$



$$Area = \frac{1}{2}ab\sin(C)$$
$$= \frac{1}{2}\cdot 10\cdot 12\cdot \sin(20)$$
$$= 60\sin(20)$$
$$\approx 20.52\ units.$$

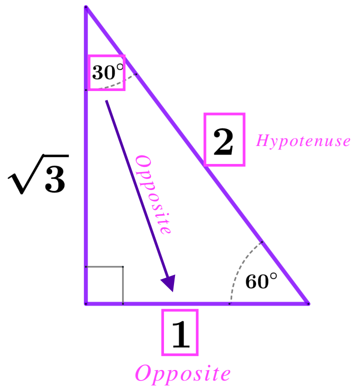
b.



$$Area = 15\,cm^2$$
$$a = 12\,cm$$
$$b = ?$$
$$\angle ab = 30^\circ$$

SOH CAH TOA

$$\sin(\theta) = \frac{Opp.}{Hyp.}$$
$$\sin(30^\circ) = \frac{1}{2}$$
$$\sin(30^\circ) = 0.5$$



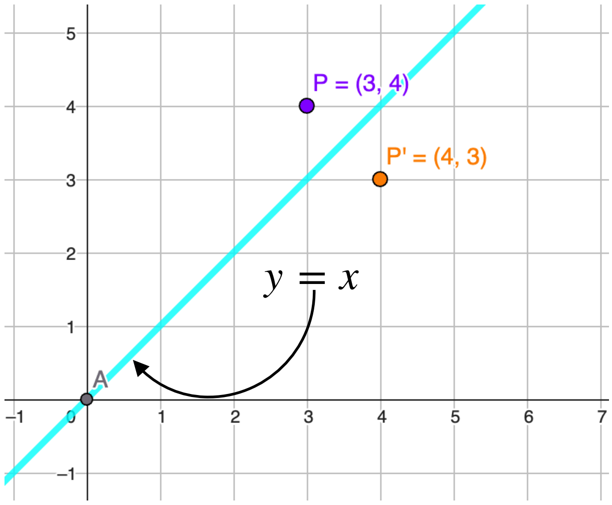
$$Area = \frac{1}{2}ab\sin(30)$$
$$15 = \frac{1}{2}\times 12b\times 0.5$$
$$15 = 6b\times 0.5$$
$$15 = 3b$$
$$\frac{15}{3} = \frac{\cancel{3}b}{\cancel{3}}$$
$$\frac{15}{3} = b$$
$$b = \frac{15}{3}$$
$$= 5$$
$$= 5\,cm.$$

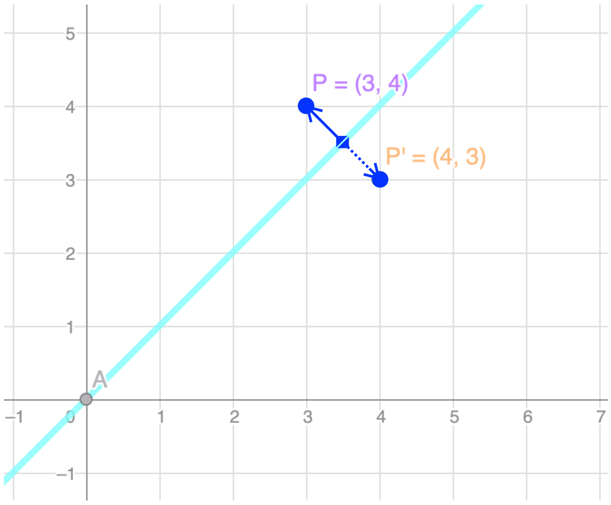
7a.

When a point is reflected over the line $y = x$, the $x-$ and $y - coordinates$ swap places.
For $P(3,\ 4)$, the $x - coordinate\ \ 3$ becomes the $y - coordinate$, and the $y - coordinate\ 4$ becomes the $x - coordinate$.
Thus, the image point is $P(3,\ 4) \Rightarrow P'(4,\ 3)$.

For a reflection over the line : $y = x$,

$$(x, y) \Rightarrow (y, x)$$





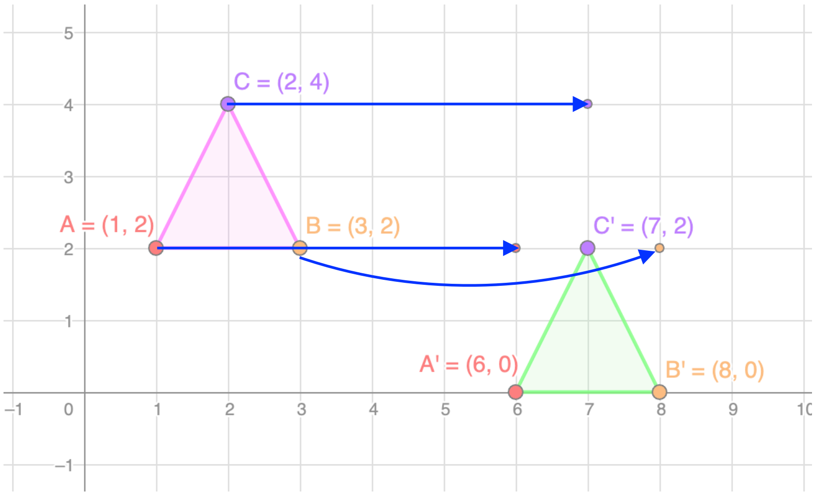
b.

A translation 5 units right adds 5 to the $x-coordinate$, and $2\ units$ down subtracts 2 from the $y-coordinate$.

For $A(1,\ 2)$:

$$x' = 1 + 5$$
$$= 6,$$
$$y' = 2 - 2$$
$$= 0,$$

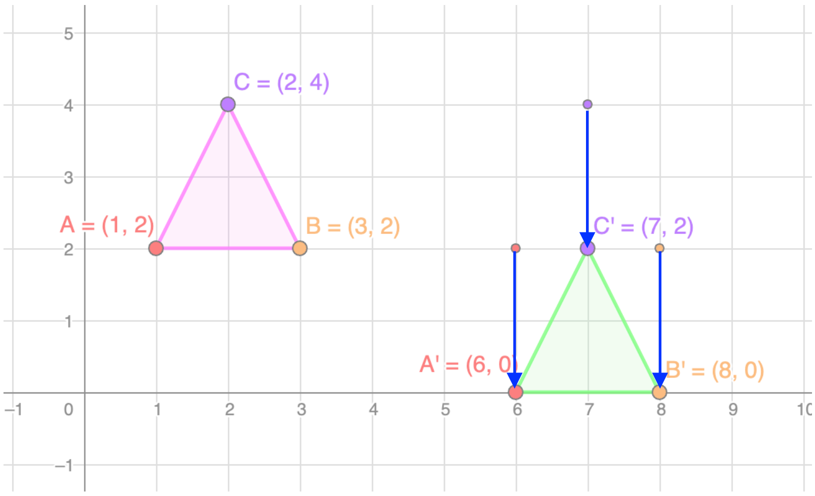
$$\rightarrow A'(6,\ 0).$$



For $B(3,\ 2)$:

$$x' = 3 + 5$$
$$= 8,$$
$$y' = 2 - 2$$
$$= 0,$$

$$\rightarrow B'(8,\ 0).$$

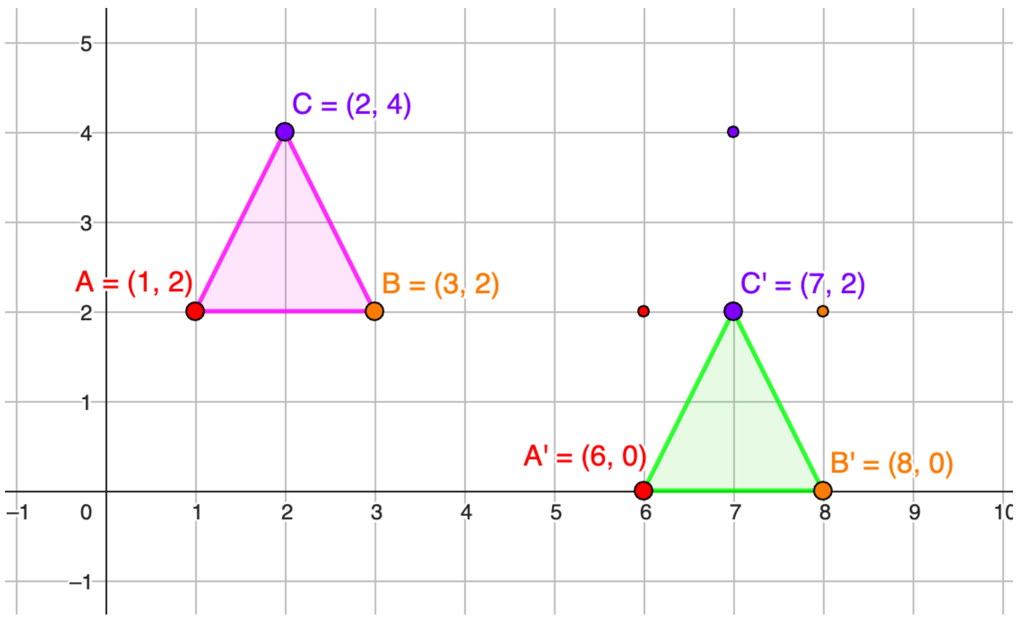


For $C(2,\ 4)$:

$$x' = 2 + 5$$
$$= 7,$$
$$y' = 4 - 2$$
$$= 2,$$

$$\rightarrow C'(7,\ 2).$$

$$A(1,\ 2),\ B(3,\ 2),\ C(2,\ 4), \Rightarrow A'(6,\ 0),\ B'(8,\ 0),\ C'(7,\ 2).$$



c.

A 90° anticlockwise rotation about the origin transforms a point $(x,\ y)$ to $(-y,\ x)$.
Apply this to each vertex:

Rotaion Coordinate Rules (around the origin)

90° counterclockwise or 270° clockwise
$(x, y) \rightarrow (-y, x)$
180° clockwise or 180° counterclockwise
$(x, y) \rightarrow (-x, -y)$
90° clockwise or 270° counterclockwise
$(x, y) \rightarrow (y, -x)$

$S(-2,\ 1)$:

$x' = -1,\ y' = -2,$
$\rightarrow S'(-1,\ -2)$.

$T(1,\ 1)$:

$x' = -1,\ y' = 1,$
$\rightarrow T'(-1,\ 1)$.

$U(1,\ -2)$:

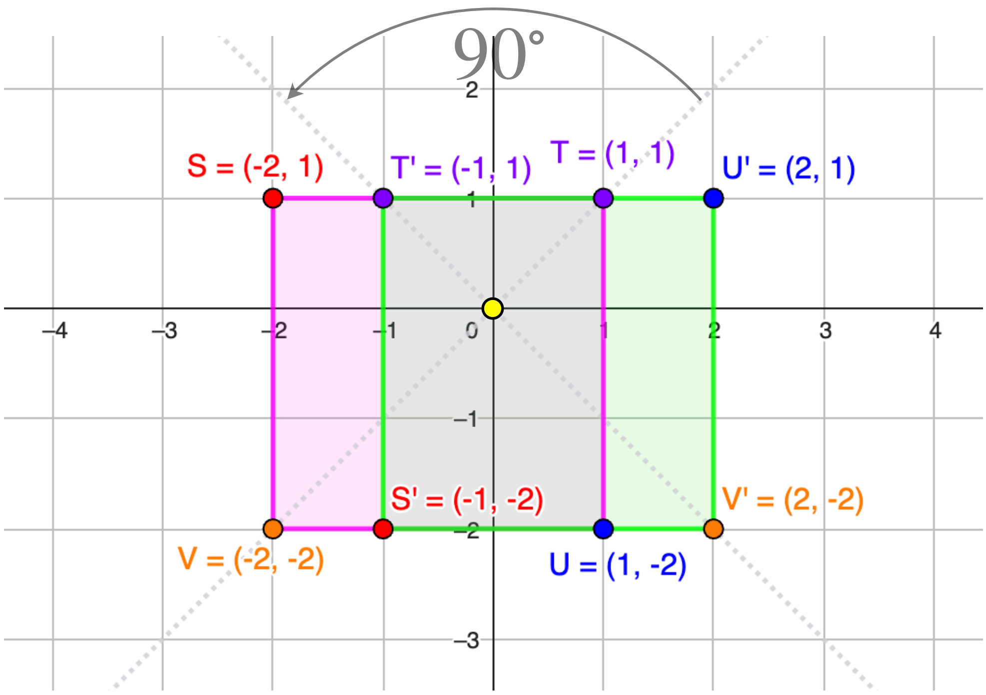
$x' = 2,\ y' = 1,$
$\rightarrow U'(2,\ 1)$.

$V(-2,\ -2)$:

$x' = 2,\ y' = -2,$
$\rightarrow V'(2,\ -2)$.

Explanation: The rotation swaps the $x-$ and $y-coordinates$, and negates the original $y-coordinate$, effectively turning the square 90° anticlockwise. The side lengths remain unchanged, but the orientation shifts, with each vertex moving to the position of its neighbour in the anticlockwise direction.

$$S(-2,\ 1),\ T(1,\ 1),\ U(1,\ -2),\ V(-2,\ -2)$$
$$\Rightarrow S'(-1,\ -2),\ T'(-1,\ 1),\ U'(2,\ 1),\ V'(2,\ -2)$$



d.

Step 1 :

Translation by $(4,\ 0)$

Add 4 to the $x-coordinate$ and 0 to the $y-coordinate$ of $(1,\ 2)$.

$$x' = 1 + 4$$
$$= 5,$$

$$y' = 2 + 0$$
$$= 2,$$
$$\rightarrow (5,\ 2).$$

Step 2 :

Reflection over $y = 0\,(x - axis)$

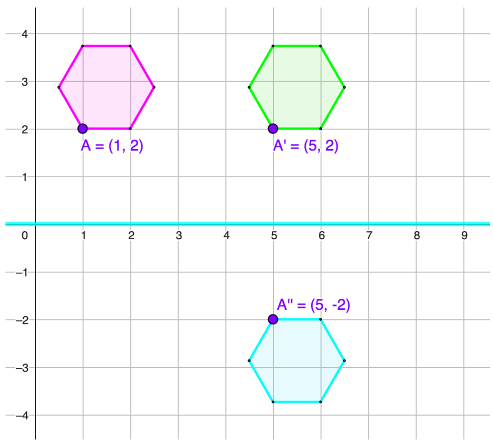
Reflecting over the $x - axis$ keeps the $x - coordinate$ the same and negates the $y - coordinate$.

For $(5,\ 2)$:

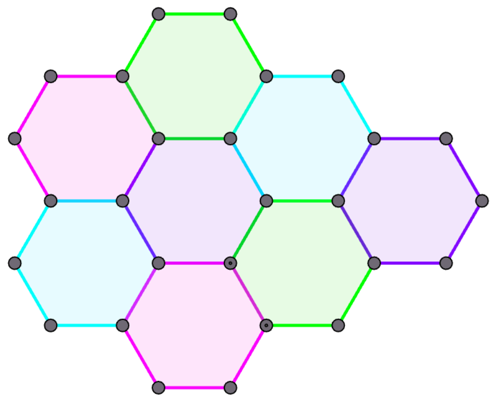
$$x' = 5,$$
$$y' = -2,$$
$$\rightarrow (5,\ -2).$$

Explanation: Tessellations require shapes to fit together without gaps or overlaps. Translating the hexagon horizontally by $(4,\ 0)$ shifts it sideways, potentially aligning it with other hexagons in a row. Reflecting over the x-axis flips it vertically, creating a mirrored version that can interlock with the original or translated hexagons. Regular hexagons tessellate naturally due to their 120° internal angles summing to 360° at shared vertices, and these transformations ensure the pattern repeats systematically across the plane.

The vertex moves to $(5, -2)$.



Tessellated Hexagons



Interesting: It is possible to colour such patterns (which includes maps with boundaries between each state), with just four colours and not have the same colour appear next to itself. This is known as the four colour theorem. Looking at the tessellation pattern above, if you focus on the purple hexagon in the middle of the diagram, it can be seen that no hexagon touching it is also purple.

Additional Notes for Teachers:

Learning Outcomes:
Students should understand and apply trigonometric ratios, solve problems involving angles of elevation and depression, and use the sine and cosine rules for non-right angled triangles.

Teaching Strategies:
Use real-world contexts like surveying or architecture to illustrate applications. Hands-on activities with clinometers or digital tools for angle measurement can be engaging.

Assessment:
Assess through mixed problems that require choosing and applying appropriate trigonometric methods.

Resources:
Use calculators with trigonometric functions, geometry software for interactive learning, or physical models for understanding angles and triangles.

This set of questions aligns with the Australian Curriculum for Year 9, aiming to develop a comprehensive understanding of trigonometry in both theoretical and practical contexts.