Ratio, Rates, Proportion, and Pythagoras

8

ACACIA TUTORING

Free and always will be!

Focus: A set of questions and solutions for Year 8 students focused on 'Ratio, Rates, and Proportion' under the "Number and Algebra" strand, tailored to the Australian Curriculum:

1. Understanding Ratios and Rates:

a) Define ratio and rate. How are they different?

Solution:
Ratio: A comparison of two or more quantities in terms of how many times one contains the other.
Example: $3:2$.

Rate: A ratio that compares different kinds of quantities, often time-related, like speed (km/h).

Difference: Ratios compare similar quantities, while rates compare different units.

b) Explain what a proportion is.

Solution:
A proportion is an equation stating that two ratios are equal, like:
$a:b = c:d, \quad or \quad \frac{a}{b} = \frac{c}{d}$

2. Working with Ratios:

a) Simplify the ratio $15:20$.

Solution:
Divide both numbers by their greatest common divisor (5):

$$15:20$$

$$\rightarrow \frac{15}{5}:\frac{20}{5}$$

$$= 3:4.$$

b) If a recipe calls for flour and sugar in the ratio $4:1$, how much sugar is needed if you use $200\,grams$ of flour?

Solution:
Since the ratio is $4:1$, for every 4 parts flour, there's 1 part sugar.

Sugar needed:

$$\rightarrow \frac{1}{4}\times 200$$
$$= \frac{200}{4}$$
$$= (\frac{20}{4})0$$
$$= 50\,grams.$$

3. Rates:

a) If a car travels $180\,km$ in $3\,hours$, what is its average speed in km/h?

Solution:
Speed $=$ Distance $\div$ Time

$$= \frac{180\,km}{3\,h}$$
$$= 60\ km/h.$$

b) If you earn $\$90$ for $6\,hours$ of work, what is your hourly rate?

Solution:
Hourly rate $=$ Earnings $\div$ $Hours$

$$= \frac{\$90}{6\ hours}$$
$$= \$15\ per\ hour.$$

$$= (15\ \$/h)$$

4. Proportions:

a) Solve the proportion: $\frac{3}{5} = \frac{x}{20}$.

Solution:

Multiply to get numbers off bottom of fractions, then solve for x:

$$\frac{3}{5} = \frac{x}{20}$$
$$\frac{3}{\cancel{5}} \times \cancel{5} = \frac{x}{20} \times 5$$
$$3 = \frac{5x}{20}$$
$$3 \times 20 = \frac{5x}{\cancel{20}} \times \cancel{20}$$
$$60 = 5x$$
$$\frac{60}{5} = \frac{\cancel{5}x}{\cancel{5}}$$
$$12 = x$$
$$x = 12.$$

OR

$$\frac{3}{5} \times 20 = \frac{x}{\cancel{20}} \times \cancel{20}$$
$$\frac{3 \times 20}{5} = x$$
$$\frac{60}{5} = x$$
$$12 = x$$
$$x = 12.$$

b) If 5 apples cost $\$2$, how much would 15 apples cost?

Solution:

Set up the proportion, then solve for x :

$$\frac{5}{2} = \frac{15}{x}$$

$$\frac{5}{\cancel{2}} \times \cancel{2} = \frac{15}{x} \times 2$$

$$5 = \frac{30}{x}$$

$$5 \times x = \frac{30}{\cancel{x}} \times \cancel{x}$$

$$5x = 30$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{30}{5}$$

$$x = \frac{30}{5}$$

$$x = \$6 .$$

OR

$$\frac{5}{2} = \frac{15}{x}$$

Cross Multiply:

$$5x = 30$$

Divide by 5 :

$$x = \$6 .$$

5. Practical Applications:

a) A map scale is $1 : 50{,}000$. How many $kilometres$ does $2\ cm$ on the map represent?

Solution:

Scale $= 1 : 50{,}0000$.

So, $1\ cm$ on the map $= 50{,}000\ cm$ in reality.

Convert cm to km :

$(1\ km = 1{,}000\ \cancel{m} \times 100\ cm / \cancel{m} = 100{,}000\ cm)$

or

$(100{,}000\ cm = 1\ km)$

$$\frac{50\cancel{,000}\ \cancel{cm}}{100\cancel{,000}\ \cancel{cm}/km} = 0.5\ km$$

$$\frac{5 \div 5}{10 \div 5}\ km = \frac{1}{2}\ km = 0.5\ km$$

So $1\ cm$ on he map $= 0.5\ km$ in real-life.

Therefore,

$$1\ cm \times 2 = 0.5\ km \times 2$$

$$2\ cm = 1\ km .$$

So, $2\ cm$ on the map $= 1\ km$ in real-life.

b) If a printer can print 15 pages in $3\,minutes$, how long will it take to print 60 pages?

Solution:

$$\frac{15}{3} = \frac{60}{x}$$

$$5 = \frac{60}{x}$$

$$5 \times x = \frac{60}{\cancel{x}} \times \cancel{x}$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{60}{5}$$

$$x = 12\,minutes.$$

OR

Set up the proportion for time, then cross-multiply:

$$\frac{15}{3} = \frac{60}{x}$$

$$15x = 180$$

$$x = \frac{180}{15}$$

$$= 12\,minutes.$$

6. Unit Rates:

a) Find the unit price if $3\,kilograms$ of apples cost $\$4.50$.

Solution:

Unit price $=$ Total cost $\div$ Total weight

$$= \frac{\$4.50}{3\,kg}$$

$$= \$1.50\ per\,kilogram.$$

$$(= 1.50\,\$/kg.)$$

b) A car uses $30\,litres$ of fuel to travel $450\,km$. What is the fuel efficiency in $km\ per\ litre$?

Solution:

Efficiency $= Distance \div$ Fuel used

$$= \frac{450\,km}{30\,L}$$

$$= 15\,km/L.$$

7. Comparing Rates:

Which is the better deal: $4\,kg$ **of rice for** $\$5$ **or** $5\,kg$ **for** $\$6$ **?**

Solution:
Calculate unit prices:

$4\,kg$ for $\$5$

$$= \frac{\$5}{4\,kg}$$

$$= \$1.25\, per\, kg\,.$$

$5\,kg$ for $\$6$

$$= \frac{\$6}{5\,kg}$$

$$= \$1.20\, per\, kg\,.$$

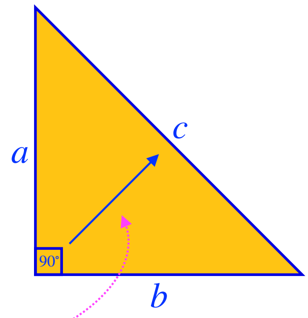
$5\,kg$ for $\$6$ is the better deal because it has a lower unit price.

8. Understanding the Pythagorean Theorem:

a) State the Pythagorean Theorem and explain what it means.

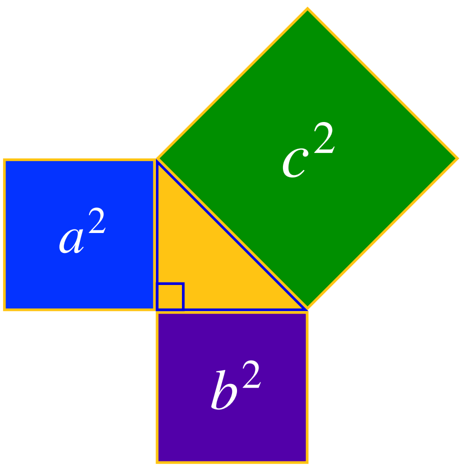
Solution:

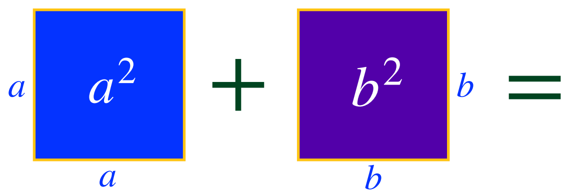
The Pythagorean Theorem states that in a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides.

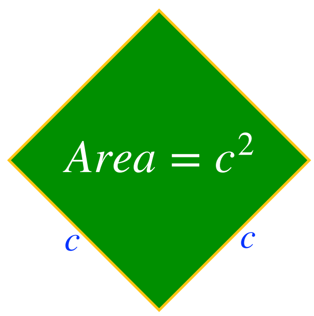


In formula terms: $a^2 + b^2 = c^2$, where c is the hypotenuse.

Note: The hypotenuse (c) is always across from the right angle.

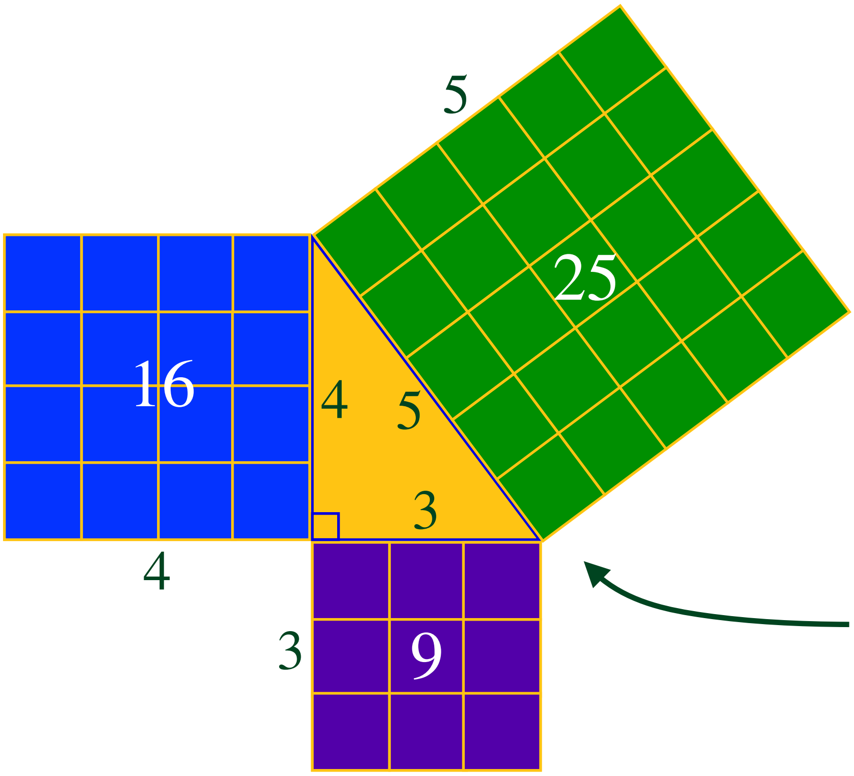






The Pythagorean Theorem, visually, describes how in a right-angled triangle, the area of blue square plus the area of the purple square always equals the area of the green square.

See another example on the following page.



Here, an example of a right-angled triangle with sides 3, 4 and hypotenuse 5.

It can be seen that the area of the green square = $5 \times 5 = 25$, which equals the area of the two sides,
i.e. the blue square = $4 \times 4 = 16$,
and the purple square = $3 \times 3 = 9$.

$\rightarrow 9 + 16 \equiv 25$

$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 \equiv 25$$

$\equiv$ means, 'the same as' or 'equivalent'.

b) How can you determine if a triangle is right-angled using this theorem?

Solution:

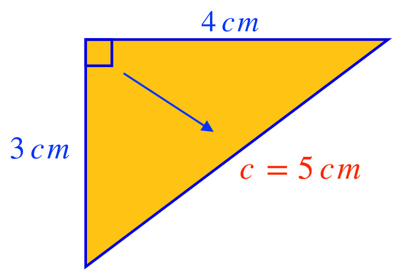
If the square of one side (the longest side or potential hypotenuse) equals the sum of the squares of the other two sides, the triangle is right-angled. Plug in values for the two sides and check they equal the square of the hypotenuse. (More examples further on, see Question 6 Converse of the Pythagorean Theorem, also Question 4 Pythagorean Triplets).

9. Finding the Hypotenuse:

a) Find the length of the hypotenuse in a right triangle where the two sides are $3\,cm$ and $4\,cm$, respectively.

Solution:

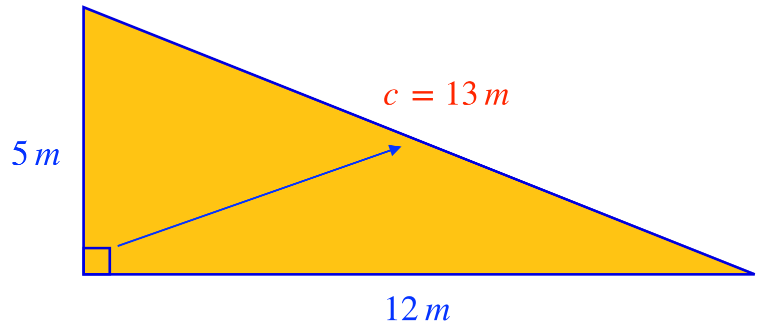
$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = c^2$$
$$9 + 16 = c^2$$
$$25 = c^2$$
$$c^2 = 25$$
$$\cancel{\sqrt{c^2}} = \sqrt{25}$$
$$c = \sqrt{25}$$
$$c = 5\,cm\,.$$



b) Calculate the hypotenuse of a right triangle if one side is $5\,m$ and the other side is $12\,m$.

Solution:

$$c^2 = a^2 + b^2$$
$$c^2 = 5^2 + 12^2$$
$$c^2 = 25 + 144$$
$$c^2 = 169$$
$$\cancel{\sqrt{c^2}} = \sqrt{169}$$
$$c = \sqrt{169}$$
$$c = 13\,m\,.$$

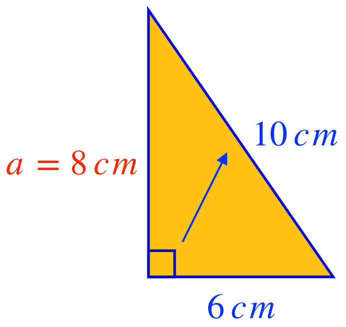


10. Finding a Side:

a) In a right triangle, the hypotenuse is $10\,cm$ and one side is $6\,cm$, what is the length of the other side?

Solution:

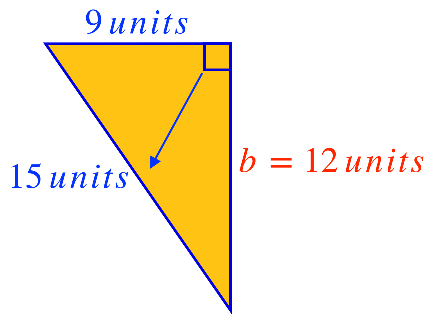
$$a^2 + b^2 = c^2$$
$$a^2 + 6^2 = 10^2$$
$$a^2 + 36 = 100$$
$$a^2 \cancel{+36} \cancel{-36} = 100 - 36$$
$$a^2 = 64$$
$$\cancel{\sqrt{a^2}} = \sqrt{64}$$
$$a = \sqrt{64}$$
$$a = 8\,cm\,.$$



b) One side of a right triangle is 9 units, and the hypotenuse is 15 units. What is the length of the other side?

Solution:

$$a^2 + b^2 = c^2$$
$$9^2 + b^2 = 15^2$$
$$81 + b^2 = 225$$
$$\cancel{81} + b^2 \cancel{-81} = 225\ -81$$
$$b^2 = 144$$
$$\cancel{\sqrt{b^2}} = \sqrt{144}$$
$$b = 12 \text{ units}\,.$$



11. Pythagorean Triplets:

a) What are Pythagorean triplets? Give an example.

Solution:

Pythagorean triplets are sets of three positive integers a, b, and c that satisfy the Pythagorean theorem, $a^2 + b^2 = c^2$. Visually, whole numbers, that make a right angled triangle. *See Question 2a. for diagram.*

An example is $(3, 4, 5)$ because $3^2 + 4^2 = 5^2$.

b) Check if $(5, 12, 13)$ is a Pythagorean triplet.

Solution:

See Question 2b. for diagram.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= 13^2 \\ 25 + 144 &= 169 \\ 169 &\equiv 169 \end{aligned}$$

$\equiv$ means, 'the same as' or 'equivalent'.

Since $5^2 + 12^2 = 13^2$, $(5, 12, 13)$ is indeed a Pythagorean triple .

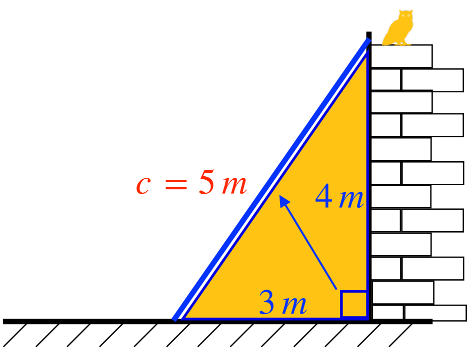
12. Practical Applications:

a) A ladder is leaning against a wall. The base of the ladder is $3\,metres$ away from the wall, and the ladder reaches $4\,metres$ up the wall. How long is the ladder?

Solution:

The ladder forms the hypotenuse of a right triangle where one side is $3\,m$ and the other is $4\,m$.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ c^2 &= 25 \\ \cancel{\sqrt{c^2}} &= \sqrt{25} \\ c &= 5\,metres\,. \end{aligned}$$

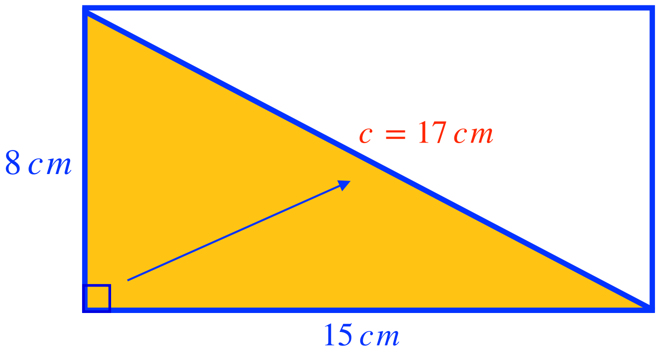


b) Find the diagonal length of a rectangle with sides $8\,cm$ and $15\,cm$.

Solution:

The diagonal forms the hypotenuse of a right triangle with the sides as sides:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 8^2 + 15^2 &= c^2 \\ 64 + 225 &= c^2 \\ 289 &= c^2 \\ c^2 &= 289 \\ \cancel{\sqrt{c^2}} &= \sqrt{289} \\ c &= 17\,cm\,. \end{aligned}$$



13. Converse of the Pythagorean Theorem:

a) Explain the converse of the Pythagorean Theorem.

Solution:

The converse states that if $a^2 + b^2 = c^2$ for the sides of a triangle, then the triangle is right-angled with c as the hypotenuse.

b) Use the converse to determine if a triangle with sides $7\,cm$, $24\,cm$, and $25\,cm$ is right-angled.

Solution:

$$a^2 + b^2 = c^2$$

Check if: Left Hand Side = Right Hand Side

$$\text{LHS} = 7^2 + 24^2 = 49 + 576$$
$$= 625$$

$$\text{RHS} = 625\,.$$

$$\text{Since LHS} \equiv \text{RHS}$$
$$25^2 \equiv 625$$

≡ means, 'the same as' or 'equivalent'.

Therefore, since $7^2 + 24^2 \equiv 25^2$, the triangle is right-angled.

Additional Notes for Teachers:

Learning Outcomes: Students should be able to interpret, simplify, and solve problems involving ratios, rates, and proportions, applying these concepts in practical contexts.

Teaching Strategies: Use real-life examples like cooking, shopping, or travel to make concepts relatable. Encourage the use of dimensional analysis or unit conversion to solve rate problems. Promote discussions on the best value or efficiency when comparing rates.

Assessment: Evaluate through tasks requiring students to solve for unknowns in proportions, calculate rates, and apply these principles to everyday scenarios.

Resources: Use kitchen scales for ratio activities, maps for scale problems, or financial calculations for rate applications.

This question set aligns with the Australian Curriculum for Year 8, focusing on the proficiencies of understanding, fluency, problem-solving, and reasoning in number and algebra, specifically in the context of ratio, rates, and proportion.