Real Numbers, Indices and Surds

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Focus: A set of questions and solutions for Year 9 students focused on 'Real Numbers, Indices, and Surds'' under the "Number and Algebra" strand, tailored to the Australian Curriculum":

$\mathbb{R}$ = *Set of Real Numbers* = $\{\mathbb{N}^*, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}\}$

$\mathbb{N}^*$ = Natural numbers,	$\{1, 2, 3 \ldots\}$
$\mathbb{N}$ = Natural numbers including zero,	$\{0, 1, 2, 3 \ldots\}$
$\mathbb{Z}$ = Integers,	$\{\ldots -2, -1, 0, 1, 2 \ldots\}$
$\mathbb{Q}$ = Rational numbers,	$\{\ldots \frac{-3}{5}, \frac{-1}{2}, \frac{-2}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5} \ldots\}$
$\mathbb{I}$ = Irrational numbers,	$\{\ldots -\pi, -\sqrt{3}, -\psi, -\sqrt{2}, \sqrt{2}, \psi, \sqrt{3}, \pi \ldots\}$

1. Understanding Real Numbers

a) Define what real numbers are.

Solution:
Real numbers are all numbers on the number line including all rational numbers (integers, fractions, and terminating or repeating decimals) and irrational numbers (non-repeating, non-terminating decimals like $\sqrt{2}$ or π).

b) How do real numbers differ from rational numbers?

Solution:
Rational numbers can be expressed as a ratio of two integers (like $\frac{1}{2}$ or 0.75), whereas real numbers also include irrational numbers, which cannot be expressed as simple fractions (like $\sqrt{3}$ or π).

c) In which set of numbers do indices NOT fit into? *(Tricky!)*

Solution:
Simply put (more complexity / accuracy comes in higher grades), $\mathbb{I}$ & $*\mathbb{N}$, Irrational numbers can't be factored & no power gives a solution of zero i.e. $a^b \neq 0$.

Indices can fit into: $\mathbb{N}\,(2^3 = 8),\ \mathbb{Z}\,(-2^3 = -8),\ \mathbb{Q}\,(5 \times 2^{-3} = \frac{5}{8}),$.

2. Classifying Numbers

Classify each of the following numbers: $5, \frac{2}{3}, 0.333..., \sqrt{16}, \sqrt{20}$ **.**

Solution:

This symbol means 'in' or 'belongs to' .

5 **:** Integer, rational, real. $\mathbb{Z} \in (\mathbb{R})$

$\frac{2}{3}$**:** Fraction, rational, real. $\mathbb{Q} \in (\mathbb{R})$

$0.333...$ **:** Repeating decimal, rational, real. $\mathbb{Q} \in (\mathbb{R})$

$\sqrt{16}$ **:** ($= 4$) Integer, real. $\mathbb{Z} \in (\mathbb{R})$

$\sqrt{20}$**:** Irrational, real. (Since $\sqrt{20}$ is approximately 4.472135955... and does not terminate or repeat). $\mathbb{I} \in (\mathbb{R})$

3. Operations with Real Numbers

a) Add: $3 + \sqrt{2} + \sqrt{2}$ **.**

Solution:

Combine like terms:

$$\begin{aligned} 3 + \sqrt{2} + \sqrt{2} &= 3 + 2 \times \sqrt{2} \\ &= 3 + 2\sqrt{2} . \end{aligned}$$

b) Multiply: $5 \times \sqrt{3}$ **.**

Solution:

$5\sqrt{3}$ (since you can't simplify further without knowing the exact value of $\sqrt{3}$).

c) Rationalise the denominator of $\frac{1}{\sqrt{5}}$ **.**

Solution:

Multiply numerator and denominator by $\sqrt{5}$:

Remember, $\frac{\sqrt{5}}{\sqrt{5}} = 1$, so we are multiplying by 1, which doesn't change the original expression , it only makes it look different .

$$= \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{1 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}}$$

$$= \frac{\sqrt{5}}{5} .$$

$\sqrt{5} \times \sqrt{5} = 5$, because:

$$\begin{aligned} \sqrt{5} \times \sqrt{5} &= 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} \\ &= (5 \times 5)^{\frac{1}{2}} \\ &= (25)^{\frac{1}{2}} \\ &= \sqrt{25} \\ &= 5 . \end{aligned}$$

4. Comparing and Rounding Real Numbers

a) Which is larger, $\sqrt{7}$ or $\sqrt{8}$?

Solution:
Since $7 < 8$,

$\rightarrow \sqrt{7} < \sqrt{8}$.

b) Order from smallest to largest: $0.5,\ \sqrt{2},\ 1.4,\ \dfrac{3}{2}$ **.**

Solution:
Convert to decimal where possible for comparison:

$$0.5,\ \sqrt{2} \approx 1.414,\ 1.4,\ \frac{3}{2} = 1.5$$

Order: $0.5,\ 1.4,\ \sqrt{2},\ 1.5$.

c) Round 3.14159 **to** 3 **decimal places (thousandths).**

Solution:
$\pi \approx 3.14159$ 9 close to 10 so round 5 up to $6 \uparrow$
$= 3.1416$ 6 close to 10 so round 1 up to $2 \uparrow$
$= 3.142$. Rounded to three decimal places.

d) Round 2.718 **to** 1 **decimal place (tenths).**

Solution:
$e \approx 2.718$ 8 close to 10 so round 1 up to $2 \uparrow$
$= 2.72$ 2 close to 0 so round 7 down to stay at $7 \downarrow$
$= 2.7$. Rounded to one decimal place.

Rounding Rules:
If 5 and above , round up ↑
If 4 and below , round down ↓

e) Convert $0.333\dot{3}$ **to a fraction.**

Solution
Let $x = 0.3333\ldots$
$= 0.\dot{3}$:

$\rightarrow 10x = 3.333\ldots$ Choose 1 decimal place, i.e. $0.\dot{3}$ so do: $x \times 10$

Now,

$$\begin{aligned} \rightarrow 9x &= 10x - x \\ &= 3.3333... - 0.3333\ldots \\ &= 3\,, \end{aligned}$$

$$\begin{aligned} \rightarrow 9x &= 3 \\ \frac{\cancel{9}x}{\cancel{9}} &= \frac{3}{9} \\ x &= \frac{3}{9} \\ &= \frac{3 \div 3}{9 \div 3} \\ &= \frac{1}{3}\,, \end{aligned}$$

$$0.333\dot{3} = \frac{1}{3}\,.$$

f) Convert $1.35\overline{35}$ to a fraction.

Solution:

Let $x = 1.353535...$

$= 1.\overline{35}$

$\rightarrow 100x = 135.\overline{35}$ Choose 2 decimal places, so do: $x \times 100$

Now,

$$\rightarrow 99x = 100x - x$$
$$= 135.\overline{35} - 1.\overline{35}$$
$$= 134,$$

$$\rightarrow 99x = 134$$
$$\frac{\cancel{99}x}{\cancel{99}} = \frac{134}{99}$$
$$x = \frac{134}{99},$$
$$1.\overline{35} = \frac{134}{99}.$$

5. Real Numbers in Context

Explain how real numbers are used in real life, giving an example.

Solution:

Real numbers are used in measurements where precision is needed. For example, the dimension of a piece of wood might be measured as $2.45\, metres$, or the temperature might be $21.7^{\circ}C$, both of which are real numbers.

6. Irrational Numbers

Irrational Numbers:

$\pi = 3.14159... \Rightarrow$ Pi

$\phi\ (\varphi) = 1.61803... \Rightarrow$ Phi - Golden Ratio

$e = 2.71828... \Rightarrow$ Euler's Number

What makes a number irrational?

Solution:

A number is irrational if it cannot be expressed as a simple fraction (ratio of integers) and its decimal representation neither terminates nor repeats.

7. Practical Application

A circle has a radius of $\sqrt{7}\ cm$. Calculate its area in exact form.

Solution:

Area of Circle $= \pi r^2$

$$= \pi \times (\sqrt{7})^2$$
$$= \pi \times (7^{\frac{1}{2}})^2$$
$$= \pi \times 7$$
$$= 7\pi\ cm^2.$$

8. Approximating Irrational Numbers

Between which two consecutive integers does $\sqrt{30}$ lie?

Solution:
Since $5^2 = 25$ and $6^2 = 36$,
$\sqrt{30}$ lies between 5 and 6 .

9. Extension Question:

Prove that $\sqrt{5}$ is irrational . *(Extension Question, Senior Specialist Mathematics)*

Solution:
Proof by contradiction *(Senior Specialist Mathematics)*:

Assume $\sqrt{5}$ isn't irrational i.e.,
It can be broken into a fraction of two rational numbers:

$\rightarrow \sqrt{5} = \dfrac{a}{b}$, where a and b are integers with no common factors other than 1.

$$\sqrt{5} \times b = \frac{a}{\cancel{b}} \times \cancel{b}$$

$$\sqrt{5}b = a\ .$$

Squaring both sides gives:

$$5b^2 = a^2\ ,$$

Meaning a^2 must be divisible by 5.
Hence, (a) must be divisible by 5.

Let $a = 5k$, then:

$$a^2 = (\ 5k\ )^2$$

$$\&\ a^2 = 5b^2$$

$$\rightarrow 5b^2 = (5k)^2$$

$$5b^2 = 25k^2$$

$$\frac{\cancel{5}b^2}{\cancel{5}} = \frac{25k^2}{5}$$

$$b^2 = \frac{25}{5}k^2$$

$\rightarrow b^2 = 5k^2$, so (b) must also be divisible by 5.

This contradicts the assumption that a and b have no common factors other than 1,

thus proving $\sqrt{5}$ is irrational .

10. Understanding Indices

a) What are indices, and how do they affect numbers?

Solution:
Indices (or exponents) indicate how many times a number (the base) is multiplied by itself. For example, 3^4 means 3 is multiplied by itself 4 times: $3 \times 3 \times 3 \times 3 = 81$.
They can significantly increase or decrease the magnitude of numbers.

b) State the laws of indices for multiplication, division, power of zero and raising a power to a power.

Solution:

Multiplication:	$a^m \times a^n = a^{m+n}$
Division:	$a^m \div a^n = a^{m-n}$
Power of a Power:	$(a^m)^n = a^{m \times n}$
Negative Power:	$a^{-m} = \dfrac{1}{a^m}$
Rational Power:	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = a^{\frac{1}{n} \times m} = \left(\sqrt[n]{a}\right)^m = a^{m \times \frac{1}{n}} = \sqrt{(a^m)}$
Zero Power:	$a^0 = 1$

11. Simplifying Expressions with Indices

a) I. Simplify $2^3 \times 2^4$ **.**

Solution:
Using the multiplication law:
$2^3 \times 2^4 = 2^{3+4}$
$= 2^7 \qquad (2^2 \times 2^2 \times 2^2 \times 2^1) = (4 \times 4 \times 4 \times 2) = (16 \times 4 \times 2)$
$= 128 . \qquad = (64 \times 2)$

II. Simplify $4 \times 3^2 \times 3$ **.**

Solution:
Using the multiplication law:
$4 \times 3^2 \times 3^1 = 4 \times 3^{2+1}$
$= 4 \times 3^3 \quad (4 \times 3 \times 3 \times 3)$
$= 4 \times 27 \quad = (12 \times 9)$
$= 108 .$

b) Simplify $\dfrac{5^7}{5^3}$.

Solution:
Using the division law:

$$\begin{aligned}\frac{5^7}{5^3} &= 5^{7-3}\\ &= 5^4\\ &= 5^2 \times 5^2\\ &= 25 \times 25 \quad (25 \times 10 + 25 \times 10 + 25 \times 5)\\ &= 625\,. \qquad = (250 + 250 + 125)\end{aligned}$$

c) Simplify $(3^2)^3$.

Solution:
Using the power of a power law:

$$\begin{aligned}(3^2)^3 &= 3^{2\times 3}\\ &= 3^6\\ &= 3^2 \times 3^2 \times 3^2\\ &= 9 \times 9 \times 9\\ &= 81 \times 9 \quad ((81 \times 10) - 81)\\ &= 729\,. \qquad = (810 - 81)\end{aligned}$$

12. Negative and Fractional Indices

a) Explain what negative indices mean. Simplify 2^{-3}.

Solution:
Negative indices mean the reciprocal of the positive index.

$$\begin{aligned}2^{-3} &= \frac{1}{2^3}\\ &= \frac{1}{8}\,.\end{aligned}$$

b) What does a fractional exponent like $4^{1/2}$ represent? Simplify it.

Solution:
A fractional exponent, $a^{1/b}$, means the b-th root of a: ($\sqrt[b]{a}$).

$$\begin{aligned}4^{1/2} &= \sqrt[2]{4}\\ &= \sqrt{4}\\ &= 2\,.\end{aligned}$$

c) Simplify $8^{2/3}$.

Solution:

$$8^{2/3} = 8^{\frac{2}{3}} = 8^{2\times\frac{1}{3}}$$
$$= 8^{\frac{1}{3}\times 2}$$
$$= (8^{1/3})^2$$
$$= (\sqrt[3]{8})^2$$
$$= (2)^2$$
$$= 4\,.$$

13. Understanding Surds

a) Define what a surd is.

Solution:

A surd is an irrational number expressed as a root (usually square or cube root) that cannot be simplified to a rational number (fraction).

Examples include $\sqrt{2}$ or $\sqrt[3]{5}$.

b) Simplify $\sqrt{50}$ to its simplest surd form.

Solution:

$$\sqrt{50} = \sqrt{25\times 2}$$
$$= \sqrt{25}\times\sqrt{2}$$
$$= 5\times\sqrt{2}$$
$$= 5\sqrt{2}\,.$$

alternatively,

$$\sqrt{50} = 50^{\frac{1}{2}}$$
$$= (25\times 2)^{\frac{1}{2}}$$
$$= 25^{\frac{1}{2}}\times 2^{\frac{1}{2}}$$
$$= \sqrt{25}\times\sqrt{2}$$
$$= 5\times\sqrt{2}$$
$$= 5\sqrt{2}\,.$$

Remember, $x^a \times y^a = (x\times y)^a$

$or\ \ (x\times y)^a = x^a\times y^a$

$E.g.\ (5\times 3)^2 = 5^2\times 3^2$

$E.g.\ 7^2\times 6^2 = (7\times 6)^2$

c) Simplify $\sqrt{75}$ to its simplest surd form.

Solution:

$$\sqrt{75} = \sqrt{25\times 3}$$
$$= \sqrt{25}\times\sqrt{3}$$
$$= 5\times\sqrt{3}$$
$$= 5\sqrt{3}\,.$$

alternatively,

$$\sqrt{75} = 75^{\frac{1}{2}}$$
$$= (25\times 3)^{\frac{1}{2}}$$
$$= 25^{\frac{1}{2}}\times 3^{\frac{1}{2}}$$
$$= \sqrt{25}\times\sqrt{3}$$
$$= 5\times\sqrt{3}$$
$$= 5\sqrt{3}\,.$$

14. Operations with Surds

a) Add: $\sqrt{8}+\sqrt{2}$ **.**

Solution:
In order to perform addition, we need to get both parts of the expression in terms of the same root, in this case: $\sqrt{2}$.

First, simplify $\sqrt{8}$,

$$\begin{aligned} &= \sqrt{4\times 2} \\ &= \sqrt{4}\times\sqrt{2} \\ &= 2\times\sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

So $\sqrt{8}=2\sqrt{2}$.

Then add,

$$\begin{aligned} \sqrt{8}+\sqrt{2} &= 2\sqrt{2}+\sqrt{2} \\ &= 2\sqrt{2}+1\sqrt{2} \\ &= 3\sqrt{2}\,. \end{aligned}$$

Remember, a surd is just a rational index, (a power that is a fraction), so all the index laws apply to them!

b) Multiply: $\sqrt{3}\times\sqrt{12}$ **.**

Solution:

$$\begin{aligned} \sqrt{3}\times\sqrt{12} &= \sqrt{3\times 12} \\ &= \sqrt{36} \\ &= 6\,. \end{aligned}$$

c) Multiply: $\sqrt{5}\times\sqrt{5}$ **.**

Solution:

$$\begin{aligned} \sqrt{5}\times\sqrt{5} &= \sqrt{5\times 5} \\ &= \sqrt{25} \\ \sqrt{5}\times\sqrt{5} &= 5\,. \end{aligned}$$

d) Multiply: $\sqrt{16}\times\sqrt{4}$ **.**

Solution:

$$\begin{aligned} \sqrt{16}\times\sqrt{4} &= \sqrt{16\times 4} \\ &= \sqrt{64} \\ &= 8\,. \end{aligned}$$

OR

$$\begin{aligned} \sqrt{16}\times\sqrt{4} &= 4\times\sqrt{4} \\ &= 4\times 2 \\ &= 8\,. \end{aligned}$$

e) Rationalise the denominator of $\dfrac{5}{\sqrt{3}}$ **.**

Solution:

Multiply numerator and denominator by $\sqrt{3}$:

$$\frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

Remember, $\dfrac{\sqrt{3}}{\sqrt{3}} = 1$, so we are multiplying by 1, which doesn't change the original expression , it only makes it look different .

$$= \frac{5 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

Alternatively, $\sqrt{3} \times \sqrt{3} = 3$, because:

$$\begin{aligned}\sqrt{3} \times \sqrt{3} &= 3^{\frac{1}{2}} \times 3^{\frac{1}{2}} \\ &= (3 \times 3)^{\frac{1}{2}} \\ &= (9)^{\frac{1}{2}} \\ &= \sqrt{9} \\ &= 3 .\end{aligned}$$

$$\begin{aligned}&= \frac{5 \times \sqrt{3}}{3^{\frac{1}{2}} \times 3^{\frac{1}{2}}} \\ &= \frac{5\sqrt{3}}{3^{\frac{1}{2}+\frac{1}{2}}} \\ &= \frac{5\sqrt{3}}{3^1} \\ &= \frac{5\sqrt{3}}{3} .\end{aligned}$$

15. Combining Indices and Surds

a) Simplify $(2\sqrt{3})^2$ **.**

Solution:

$$\begin{aligned}(2\sqrt{3})^2 &= (2^2)(\sqrt{3})^2 \\ &= (2^2) \times 3 \\ &= 4 \times 3 \\ &= 12 .\end{aligned}$$

OR

$$\begin{aligned}(2\sqrt{3})^2 &= (2^2)(\sqrt{3})^2 \\ &= (2^2)(3^{\frac{1}{2}})^2 \\ &= (2^2)(3^{\frac{1}{2}\times 2}) \\ &= 4 \times 3^1 \\ &= 4 \times 3 \\ &= 12 .\end{aligned}$$

b) Evaluate $8^{1/3} \times 8^{2/3}$ **.** [$or \ = \sqrt[3]{8} \times \sqrt[3]{8^2}$ $or \ = (\sqrt[3]{8})^1 \times (\sqrt[3]{8})^2$ $or \ = (8^{\frac{1}{3}})^1 \times (8^{\frac{1}{3}})^2$]

Solution:

$$\begin{aligned}8^{\frac{1}{3}} \times 8^{\frac{2}{3}} &= 8^{\frac{1}{3} + \frac{2}{3}} \\ &= 8^{\frac{1+2}{3}} \\ &= 8^{\frac{3}{3}} \\ &= 8^1 \\ &= 8 .\end{aligned}$$

16. Practical Application

The length of the diagonal of a square with side length $4\sqrt{2}\,cm$ is given by side length $\times\sqrt{2}$. Calculate this diagonal length.

Solution:

$$\begin{aligned}\text{Side Length}\times\sqrt{2} &= 4\sqrt{2}\times\sqrt{2}\\ &= 4\times\sqrt{2}\times\sqrt{2}\\ &= 4\times(\sqrt{2})^2\\ &= 4\times 2\\ &= 8\,cm\,.\end{aligned}$$

OR

$$\begin{aligned}\text{Side Length}\times\sqrt{2} &= 4\sqrt{2}\times\sqrt{2}\\ &= 4\times\sqrt{2}\times\sqrt{2}\\ &= 4\times(\sqrt{2})^2\\ &= 4\times(2^{\frac{1}{2}})^2\\ &= 4\times 2^{\frac{1}{2}\times 2}\\ &= 4\times 2^{\frac{2}{2}}\\ &= 4\times 2^1\\ &= 4\times 2\\ &= 8\,cm\,.\end{aligned}$$

17. Applying Laws of Indices in Equations

a) Solve for x in the equation $3^{2x}=81$.

Solution:

$$3^{2x}=81$$

Recognise that:

$$81 = 3^4 \text{ so,}$$

$$3^{2x}=81$$

$$\rightarrow 3^{2x}=3^4,$$

Therefore:

$$2x=4$$

$$\frac{\cancel{2}x}{\cancel{2}}=\frac{4}{2}$$

$$x=2\,.$$

b) Solve for x in the equation $4^{2x}=64$.

Solution:

$$4^{2x}=64$$

Recognise that:

$$64 = 4^3 \text{ so,}$$

$$4^{2x}=64$$

$$\rightarrow 4^{2x}=4^3,$$

Therefore:

$$2x=3$$

$$\frac{\cancel{2}x}{\cancel{2}}=\frac{3}{2}$$

$$x=\frac{3}{2}\,.$$

Additional Notes for Teachers

Learning Outcomes: Students should understand the nature of real numbers, differentiate between rational and irrational numbers, perform operations with them, and use them in practical contexts. Students should be able to manipulate expressions involving indices and surds, understand the relationship between indices and roots, and apply these concepts in problem-solving.

Teaching Strategies: Use number lines to visualise where different types of real numbers fit. Employ technology or calculators to approximate irrational numbers, but stress the importance of exact forms in calculations. Discuss real-life applications where precise measurements are crucial, emphasising the use of real numbers. Use real-life contexts like scaling, areas, or volumes to apply indices and surds. Employ visual aids or software for demonstrating square roots and cube roots. Encourage students to check their simplifications by converting back to decimal form for verification.

Assessment: Assess through exercises where students classify numbers, perform operations, compare values, and solve problems involving real numbers in both theoretical and practical contexts. Evaluate through exercises requiring simplification of expressions, solving equations involving indices, and operations with surds.

Resources: Number line tools, calculators with irrational number capabilities, or software for demonstrating number properties.Calculators for checking, but emphasise understanding over reliance on technology; worksheets with varied problems for practice.

This question set aligns with the Australian Curriculum for Year 9, focusing on the proficiencies of understanding, fluency, problem-solving, and reasoning in number and algebra, specifically in the context of real numbers, indices and surds.