Equations, Inequalities, Quadratics, and Finance

9

Free and always will be!

Focus: A a set of questions and solutions for Year 9 students focused on 'Linear and Quadratic Equations' under the "Number and Algebra" strand, tailored to the Australian Curriculum:

1. Understanding Linear Equations:

a) Define what a linear equation is and how it differs from a quadratic equation.

Solution:
A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and a single variable. It has a highest power of 1 (e.g., $2x^1 + 3 = 7$) . A quadratic equation includes a term with the variable raised to the second power, and its general form is $ax^2 + bx + c = 0$.

b) Solve the linear equation $3x - 7 = 11$.

Solution:
Add 7 to both sides:

$$3x - 7 = 11$$
$$3x \cancel{-7} \cancel{+7} = 11 + 7$$
$$3x = 18,$$

Divide by 3 :

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{18}{3}$$
$$x = \frac{18}{3}$$
$$x = 6.$$

c) Solve the linear equation $-x = -3$.

Solution:
Divide by -1 on both sides:

$$\frac{\cancel{-1}x}{\cancel{-1}} = \frac{-3}{-1}$$
$$x = 3.$$

d) Solve the linear equation $-x-7=-3$**.**

Solution:
Add 7 to both sides:

$$-x-7=-3$$
$$-x\cancel{-7}\cancel{+7}=-3\ +7$$
$$-x=4,$$

Divide by -1:

$$-1x=4$$
$$\frac{\cancel{-1}x}{\cancel{-1}}=\frac{4}{-1}$$
$$x=-4.$$

e) Solve the linear inequality $3x-7>11$**.**

Solution:
Add 7 to both sides:

$$3x-7>11$$
$$3x\cancel{-7}\cancel{+7}>11\ +7$$
$$3x>18,$$

Divide by 3:

$$\frac{\cancel{3}x}{\cancel{3}}>\frac{18}{3}$$
$$x>\frac{18}{3}$$
$$x>6.$$

f) Solve the linear inequality $-3x-7>11$**.**

Solution:
Add 7 to both sides:

$$-3x-7>11$$
$$-3x\cancel{-7}\cancel{+7}>11\ +7$$
$$-3x>18,$$

Divide by -3,
Remember to reverse the inequality:

$$\frac{\cancel{-3}x}{\cancel{-3}}<\frac{18}{-3}$$
$$x<\frac{+18}{-3}$$
$$x<-6.$$

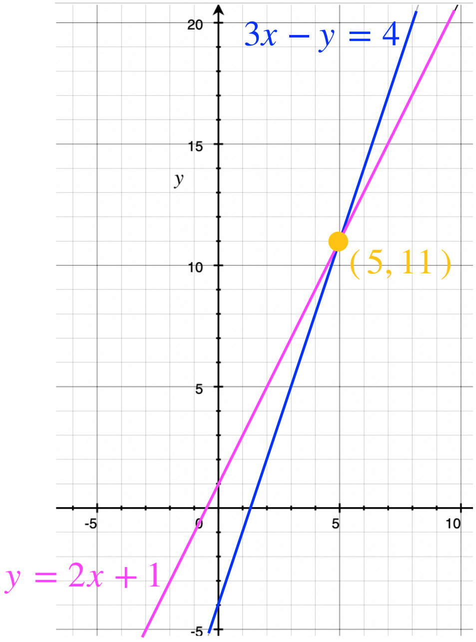
Remember, when manipulating inequalities by multiplying or dividing by a negative number, the inequality must be reversed.

2. Solving Systems of Linear Equations:

a) Solve the system of equations using substitution: $\begin{cases} y = 2x + 1 \\ 3x - y = 4 \end{cases}$.

Solution:

$$\begin{cases} y = 2x + 1 \\ 3x - y = 4 \end{cases}$$



Substitute $y = 2x + 1$ into $3x - y = 4$:

$$3x - (2x + 1) = 4$$
$$3x - 2x - 1 = 4$$
$$x - 1 = 4$$
$$x - 1 + 1 = 4 + 1$$
$$x = 5,$$

Substitute $x = 5$ into $y = 2x + 1$:

$$y = 2 \times (5) + 1$$
$$= 10 + 1$$
$$y = 11,$$

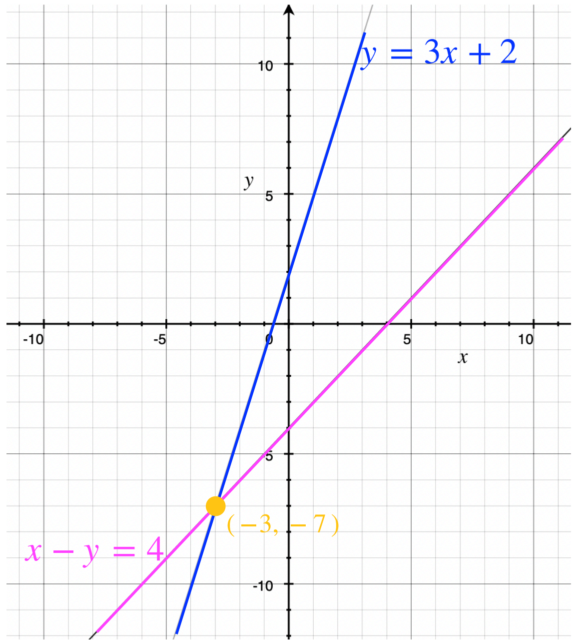
Solution: $x = 5$, $y = 11$.

b) Solve the system of equations using substitution: $\begin{cases} y = 3x + 2 \\ x - y = 4 \end{cases}$.

Solution:

$$\begin{cases} y = 3x + 2 \\ x - y = 4 \end{cases}$$

Substitute $y = 3x + 2$ into $x - y = 4$:



$$x - (3x + 2) = 4$$
$$x - 3x - 2 = 4$$
$$1x - 3x - 2 = 4$$
$$-2x - 2 = 4$$
$$-2x - 2 + 2 = 4 + 2$$
$$-2x = 6$$
$$\frac{-2x}{-2} = \frac{+6}{-2}$$
$$x = -3,$$

Substitute $x = -3$ into $y = 3x + 2$:

$$y = 3 \times (-3) + 2$$
$$= -9 + 2$$
$$y = -7,$$

Solution: $x = -3$, $y = -7$.

c) Solve the system of equations using elimination: $\begin{cases} 2x + 3y = 8 \\ 2x - y = 2 \end{cases}$.

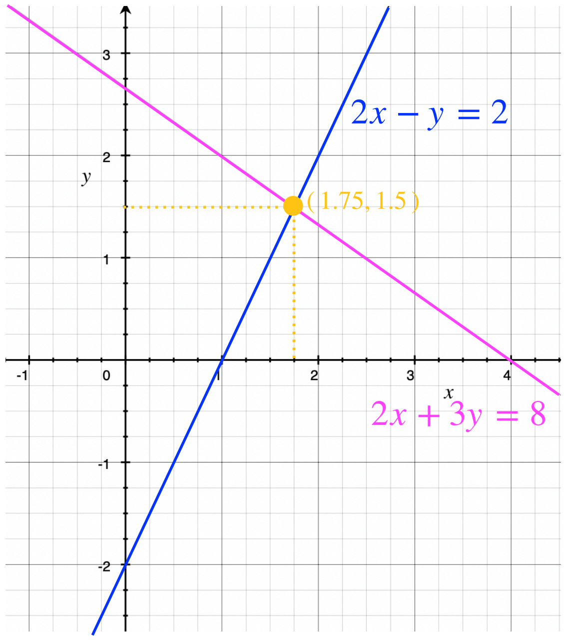
Solution:

$$\begin{cases} 2x + 3y = 8 \\ 2x - y = 2 \end{cases}$$

If signs are:
opposite → change to −
same → change to +

Subtract the second equation from the first:

$$(2x + 3y) - (2x - y) = 8 - 2$$
$$2x + 3y - 2x + y = 6$$
$$\cancel{2x} + 3y \cancel{- 2x} + y = 6$$
$$4y = 6$$
$$\frac{\cancel{4}y}{\cancel{4}} = \frac{6}{4}$$
$$y = \frac{6}{4}$$
$$y = 1.5,$$



Substitute $y = 1.5$ into $2x - y = 2$:

$$2x - 1.5 = 2$$
$$2x = 3.5$$
$$\frac{\cancel{2}x}{\cancel{2}} = \frac{3.5}{2}$$
$$x = \frac{3.5}{2}$$
$$x = 1.75,$$

Solution:

$$x = 1.75, \ y = 1.5.$$

3. Understanding Quadratic Equations:

a) What is the general form of a quadratic equation?

Solution:
The general form of a quadratic equation is:

$ax^2 + bx + c = 0$, where $a \neq 0$.

b) Identify the coefficients a, b, and c, in the quadratic equation $2x^2 - 4x + 1 = 0$.

Solution:

$$2x^2 - 4x + 1 = 0$$
$$a = +2, \ b = -4, \ c = +1.$$

4. Solving Quadratic Equations by Factorisation:

a) Solve $x^2 - 7x + 12 = 0$ by factorising.

Solution:

Look for two numbers that multiply to 12 (the constant term) and add to -7 (the coefficient of x):
Numbers are -3 and -4, so:

$$x^2 - 7x + 12$$

$$\_ \times \_ = 12 \ and \ \_ + \_ = -7$$
$$\rightarrow -3 \times -4 = 12 \ and \ -3 + -4 = -7$$

$$= (x-3)(x-4).$$
$$\rightarrow x = 3 \ and \ x = 4.$$

OR

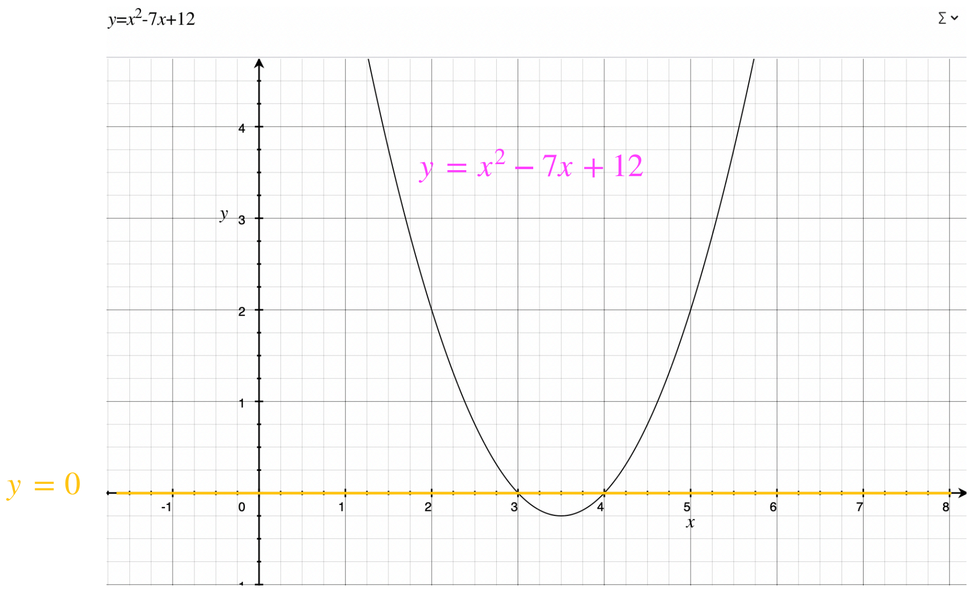
$$x^2 \ -7x \ + \ 12$$

$$\_ \times \_ = +12 \ and \ \_ + \_ = -7$$
$$\rightarrow -3 \times -4 = +12 \ and \ -3 + -4 = -7$$

$$= (x \ - \ 3)(x \ - \ 4)$$

$$= (x-3)(x-4).$$
$$\rightarrow x = 3 \ and \ x = 4.$$

Explanation of the final step on following page…



Factorise:

$$x^2 - 7x + 12 = 0$$
$$\rightarrow (x-3)(x-4) = 0$$

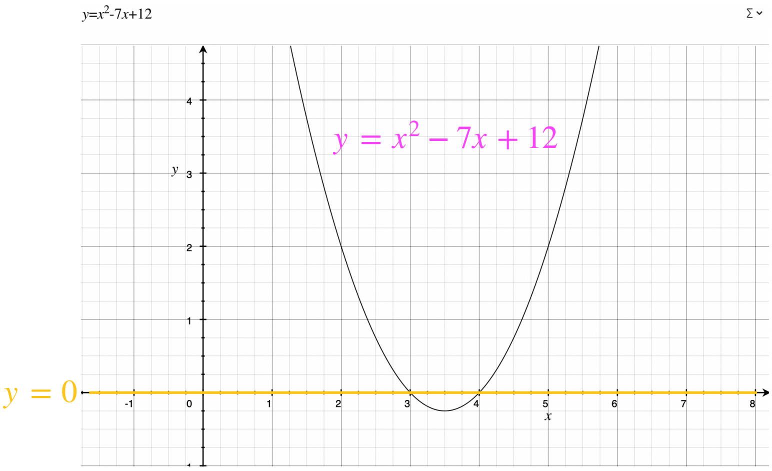
$$(x-3)(x-4) = 0$$
$$\frac{(x-3)(x-4)}{(x-4)} = \frac{0}{(x-4)} \quad , \frac{0}{(x-4)} = 0$$
$$(x-3) = 0 \, .$$

$$(x-3)(x-4) = 0$$
$$\frac{(x-3)(x-4)}{(x-3)} = \frac{0}{(x-3)} \quad , \frac{0}{(x-3)} = 0$$
$$(x-4) = 0 \, .$$

$$\rightarrow x - 3 = 0 \;\; or \;\; x - 4 = 0$$

$$x - 3 = 0$$
$$x - 3 + 3 = 0 + 3$$
$$x = 3 \, .$$

$$x - 4 = 0$$
$$x - 4 + 4 = 0 + 4$$
$$x = 4 \, .$$



This gives:

$$x = 3 \;\; or \;\; x = 4 \, .$$

b) Solve $2x^2 + 4x - 6 = 0$ **.**

Solution: Factor out the common factor (2) :

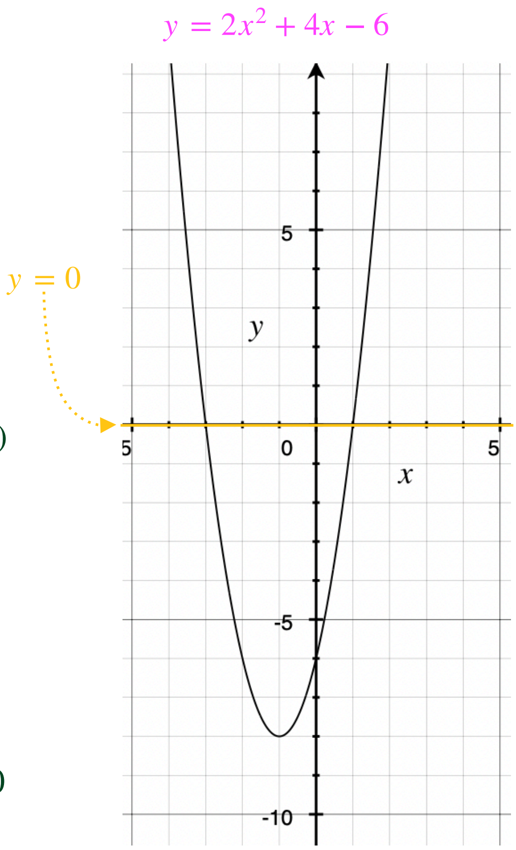
$$2x^2 + 4x - 6 = 0$$
$$2 \cdot x^2 + 2 \cdot 2x + 2 \cdot (-3) = 0$$
$$2(x^2 + 2x - 3) = 0$$

Factorise $x^2 + 2x - 3$:
Numbers are 3 and -1 ,

$$\rightarrow x^2 + 2x - 3$$
$$= (x+3)(x-1)$$

So, $2x^2 + 4x - 6 = 0$

$$\rightarrow 2(x+3)(x-1) = 0$$
$$\frac{2(x+3)(x-1)}{2} = \frac{0}{2}$$
$$(x+3)(x-1) = 0$$



Solutions:

$x + 3 = 0$ or $x - 1 = 0$

$x = -3$ or $x = 1$.

5. Solving Quadratic Equations Using the Quadratic Formula:

a) State the quadratic formula and when it is used.

Solution:

The quadratic formula is :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

Used for solving ANY quadratic equation :

$ax^2 + bx + c = 0,$

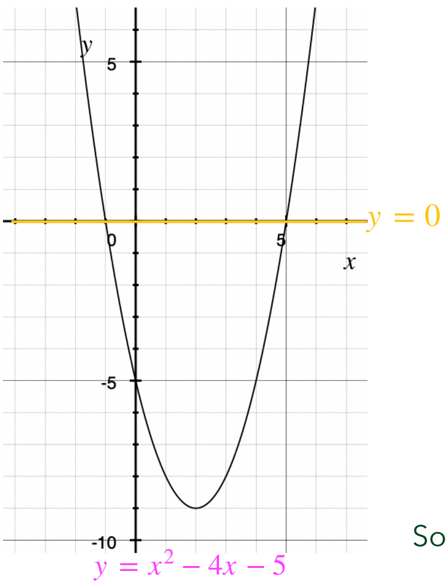
Especially when factorising is not straight forward.

b) Solve $x^2 - 4x - 5 = 0$ using the quadratic formula.

Solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

Here, $a = 1, b = -4, c = -5$:



$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1}$$

$$x = \frac{4 \pm \sqrt{16 + 20}}{2}$$

$$x = \frac{4 \pm \sqrt{36}}{2}$$

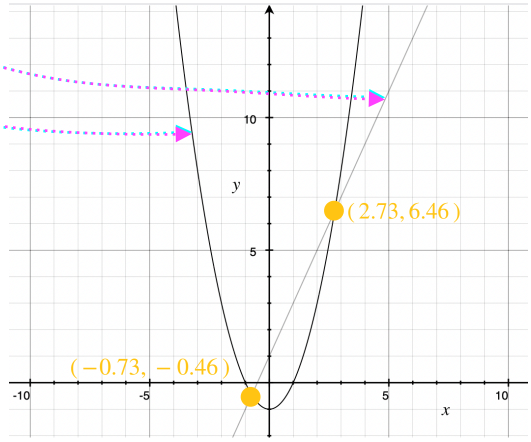
$$x = \frac{4 \pm 6}{2}$$

$$x = \frac{4 + 6}{2} = 5 \; and \; x = \frac{4 - 6}{2} = -1$$

Solutions: $x = 5 \; and \; x = -1.$

6. Solving Linear and Quadratic Equations Simultaneously:

Solve the system of equations: $\begin{cases} y = 2x + 1 \\ y = x^2 - 1 \end{cases}$.



Solution:

$$\begin{cases} y = 2x + 1 \\ y = x^2 - 1 \end{cases}$$

Since $y = y$, i.e. both equations are of the form $y = \ldots$,
Set the equations equal to each other :

$$2x + 1 = x^2 - 1$$
$$\text{Rearrange : } \cancel{2x}\cancel{-2x} + 1 = x^2 - 1 - 2x$$
$$1 = x^2 - 1 - 2x$$
$$\cancel{1}\cancel{-1} = x^2 - 1 - 2x - 1$$
$$0 = x^2 - 2x - 2$$
$$x^2 - 2x - 2 = 0,$$
$$1x^2 - 2x - 2 = 0,$$

Use the quadratic formula ($a = 1, b = -2, c = -2$) :

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1}$$
$$x = \frac{2 \pm \sqrt{4 + 8}}{2}$$
$$x = \frac{2 \pm \sqrt{12}}{2}$$
$$x = \frac{2 \pm 2\sqrt{3}}{2}$$
$$= 1 \pm \sqrt{3}.$$

Solutions :

$$x = 1 + \sqrt{3} \ and \ x = 1 - \sqrt{3}$$

For y, substitute x into $y = 2x + 1$: (could also substitute into $y = x^2 - 1$)

If $x = 1 + \sqrt{3}$, then $y = 2(1 + \sqrt{3}) + 1 = 2 + 2\sqrt{3} + 1 = 3 + 2\sqrt{3}$

If $x = 1 - \sqrt{3}$, then $y = 2(1 - \sqrt{3}) + 1 = 2 - 2\sqrt{3} + 1 = 3 - 2\sqrt{3}$

Final solutions :

$$\rightarrow (1 + \sqrt{3}, \ 3 + 2\sqrt{3}) \ and \ (1 - \sqrt{3}, \ 3 - 2\sqrt{3}).$$
$$\approx (2.73, \ 6.46) \ and \ (-0.73, \ -0.46).$$

7. Identifying Linear and Quadratic Equations:

Graph $y = x^2$ and $y = 2x - 1$:

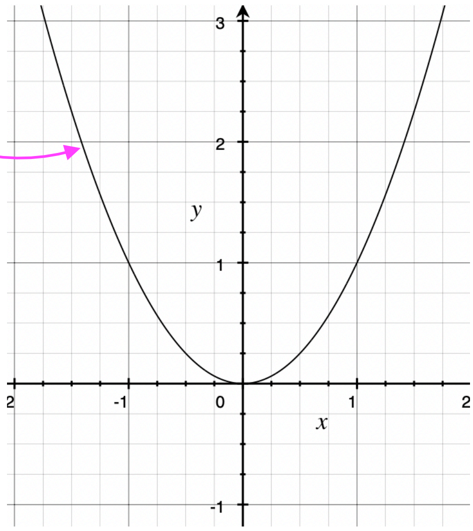
Solution

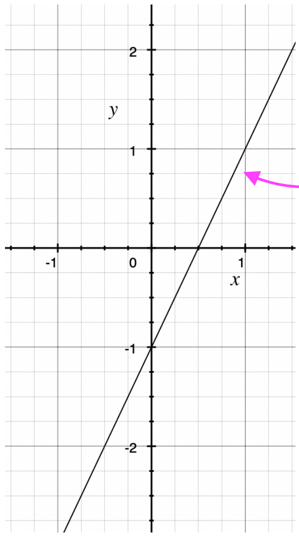
A parabola opening upwards.

$y = x^2$ represents a quadratic function with a positive coefficient $(+1)$ i.e $y = +1x^2$, hence it opens upwards.

$y = mx + c$ where $m = slope = \dfrac{rise}{run}$ and $c = y - intercept$. $y = 2x - 1$:

A straight line passing through the $y - axis$ at $y = -1$, with a slope of $\dfrac{2}{1} = \dfrac{Rise}{Run}$.





8. Sketching Graphs: Slope / Intercept form and Point / Intercept form:

a) Sketch the graph of the linear function $y = 2x + 1$. Indicate the $slope$ and $y - intercept$.

Solution:

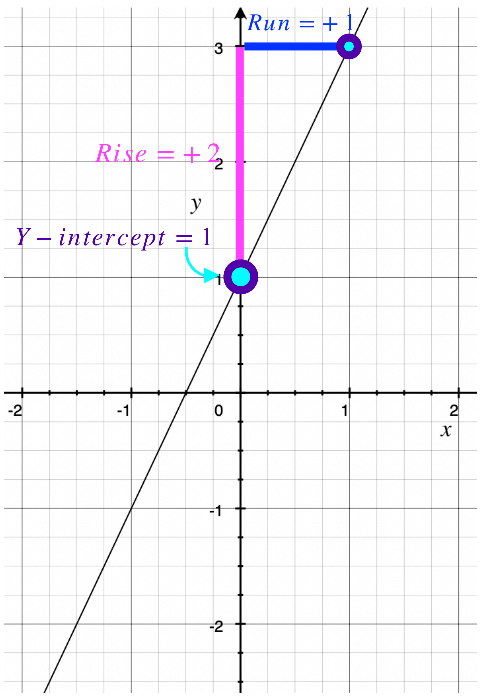
$$y = 2x + 1$$
$$y = mx + c$$

$Slope\ (m)$:

$$m = 2 = \frac{+2}{+1} = \frac{rise}{run} .$$

$y - intercept\ (c)$:

$$c = +1 .$$



[Description for drawing on graph paper with labelled axes :
Draw a point at $x = 1$, then go up 2 and across 1 , draw a point here, then draw a straight line between the two points.]

b) Given the quadratic function $y = -x^2 + 4x - 3$: Determine the vertex of the parabola and the $y - intercept$. Sketch the graph.

Solution:

Vertex, use the formula :

$$x = -\frac{b}{2a} \text{ where } a = -1 \text{ and } b = 4 :$$

$$x = -\frac{4}{2 \times (-1)}$$

$$= \frac{-4}{-2}$$

$$= \frac{\cancel{-1} \times 4}{\cancel{-1} \times 2}$$

$$= \frac{4}{2}$$

$$x = 2 .$$

Substituting $x = 2$ into the equation for y :

$$y = -(2)^2 + 4(2) - 3$$

$$= -4 + 8 - 3$$

$$y = 1 .$$

Substituting $x = 0$ into the equation for y :

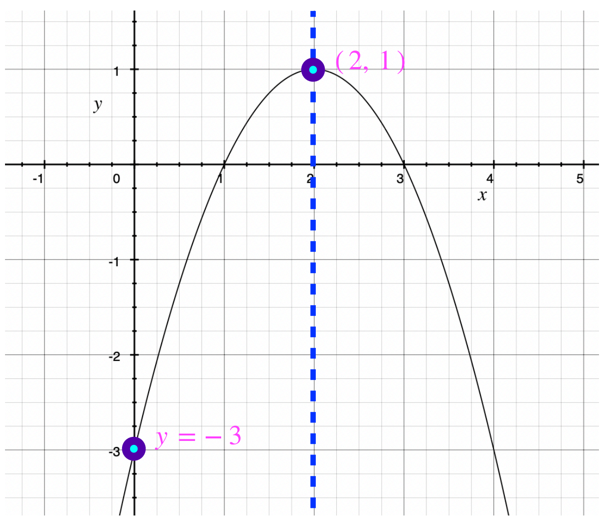
$$y = -(0)^2 + 4(0) - 3$$

$$= 0 + 0 - 3$$

$$y = -3 .$$

So, the vertex is at $(2, 1)$, and the $y - intercept$ is at $y = -3$.

[Description for sketching:
Draw a parabola opening downwards with the vertex at $(2, 1)$.
The parabola crosses the $y - axis$ at $y = -3$ (when $x = 0$)
and has symmetry about $x = 2$.]



9. Financial Applications (Simple Interest):

a) Calculate the simple interest earned on an investment of $\$2{,}500$ **at an interest rate of** 4% *per annum* **for** 3 *years*. **Show your working using the formula** $I = PRT$.

Solution:

$P = 2500$ (principal)

$R = 4\%$

$= \frac{4}{100}$

$= 0.04$ (rate as a decimal in years)

$T = 3$ (time in years)

$I = PRT$

$I = 2500 \times 0.04 \times 3$

$= 100 \times 3$

$= 300$.

The simple interest earned is $\$300$.

b) Calculate the simple interest earned on an investment of $\$108{,}000$ **at an interest rate of** 4% *per annum* **for** 9 *months*. **Show your working using the formula** $I = PRT$.

Solution:

$P = 108{,}000$ (principal)

$R = 4\%$

$= \frac{4}{100}$

$= 0.04$ (rate as a decimal per year)

Divide by 12 to convert to per month:

$\rightarrow 0.04 \div 12$

$= \frac{0.04}{12} \left(= \frac{1}{300}\right)$ (rate in months)

≈ 0.00333

$T = 9$ (time in months)

$I = PRT$

$I = 108{,}000 \times \left(\frac{0.04}{12}\right) \times 9$

$= 360 \times 9$

$= 3{,}240$.

The simple interest earned is $\$3{,}240$.

Note: You can either convert the rate to months (as shown in the example), **OR** you could convert the time to years $\left(T = \frac{9}{12} = \frac{3}{4} = 0.75 \textit{ years}\right)$.

The most important part is that both the ***rate*** and the ***time*** are in the **same units**.

→ $I = 108{,}000 \times 0.04 \times 0.75$

$= 4{,}320 \times 0.75$

$= 3{,}240$.

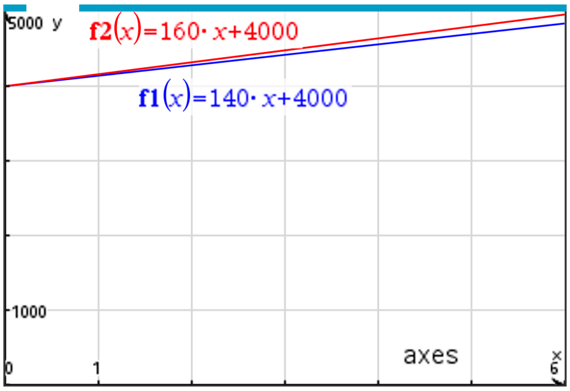
The simple interest earned is $\$3{,}240$.

c) Sarah has $\$4{,}000$ to invest. She is considering two options:

Option A: A savings account offering $3.5\ \%$ simple interest per annum for $5\ years$.
Option B: A term deposit offering $4\ \%$ simple interest per annum for $3\ years$.

I) Write a linear equation for each option, where (A) is the total amount and (T) is time in years.
II) Calculate the total amount for each option.
III) Which option gives the highest return, and by how much more?

Solution:
I) Write the linear equations: The total amount $A = P + I$, where $I = PRT$.



Option A:

$P = 4000,\ R = 3.5\ \% = 0.035,\ (T)$ is variable.

$I = 4000 \times 0.035 \times T$

$I = 140T$

$A = 4000 + 140T$. $blue\ line$

Option B:

$P = 4000,\ R = 4\ \% = 0.04,\ (T)$ is variable.

$I = 4000 \times 0.04 \times T$

$I = 160T$,

$A = 4000 + 160T$. $red\ line$

II) Calculate the total amount:

Option A:

$$T = 5\ years$$
$$A = 4000 + 140 \times 5$$
$$= 4000 + 700$$
$$A = 4700$$

Total amount $= \$4{,}700.$

Option B:

$$T = 3\ years$$
$$A = 4000 + 160 \times 3$$
$$= 4000 + 480$$
$$A = 4480$$

Total amount $= \$4{,}480$.

III) Compare and find the difference:

Option A: $\$4{,}700$
Option B: $\$4{,}480$

Difference:
$\rightarrow 4700 - 4480 = 220$.

Option A gives the highest return by $220 more than Option B.
Answer: Sarah should choose Option A, earning $220 more.

d) Jake borrows $\$1{,}200$ at $2\,\%$ simple interest $per\ \ month$. He plans to repay the loan after $4\ months$.

I) Write a linear equation for the total amount (A) owed, where (T) is time in months.
II) Calculate the total amount owing after $4\ months$.
III) If Jake can only deposit $\$1{,}300$ in one go (he doesn't want to carry change), how much more time (in months) will he need to wait to pay off the balance?

Solution:
I) Write the linear equations:

$A = P + I$, where $I = PRT$.
$P = 1200,\ R = 2\,\% = 0.02\ per\ \ month,\ (T)$ is variable.

$I = 1200 \times 0.02 \times T$
$I = 24T$, (Time in months)

$A = 1200 + 24T$. (Time in months)

II) Calculate the total amount after $4\ months$:

$T = 4$
$A = 1200 + 24T$

$A = 1200 + 24 \times 4$
$\quad = 1200 + 96$
$A = 1296$.

Total amount $= \$1{,}296$.

III) Calculate additional time needed:

Jake can only pay $\$1{,}300$.
Amount owed after $4\ months = \$1{,}296.$

Remaining balance $= 1{,}300 - 1{,}296$
$= \$4$ we need a little extra time to reach $\$1{,}300$.

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Corrected approach:

Solve for (T) when $A = 1300$. (i.e. how much time until the loan reaches $\$1{,}300$).

$$1300 = 1200 + 24T$$
$$1300 - 1200 = \cancel{1200} + 24T \cancel{-1200}$$
$$1300 - 1200 = 24T$$
$$100 = 24T$$
$$\frac{100}{24} = \frac{\cancel{24}T}{\cancel{24}}$$
$$T = \frac{100}{24}$$
$$T \approx 4.1667...\ months$$ (approx. $4.17\ months$).

Additional time beyond $4\ months$:

$4.17 - 4 = 0.17\ months$ (about $5\ days$, since $0.17 \times 30 \approx 5$).

Answer: Jake needs an extra $0.17\ months$ (approx. $5\ days$) for the loan to reach $\$1{,}300$ so he doesn't have to carry $\$4$ change.

Additional Notes for Teachers:

Learning Outcomes: Students should be able to solve linear and quadratic equations using various methods, understand the relationship between equations, and apply these concepts in solving systems of equations.

Teaching Strategies:

Use graphing tools to visualise how linear and quadratic equations intersect.

Encourage checking solutions by substituting back into original equations.

Discuss the practical applications like predicting trends or solving real-world optimisation problems.

Assessment: Evaluate through tasks requiring students to solve linear and quadratic equations, use the quadratic formula, and solve systems of equations.

Resources: Graphing calculators, algebra software, or worksheets with a mix of equation types for practice.

This question set aligns with the Australian Curriculum for Year 9, focusing on the proficiencies of understanding, fluency, problem-solving, and reasoning in number and algebra, specifically in the context of linear and quadratic equations.