# 3D Pythagoras, Spheres, Cones, and Geometric Proofs

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**Focus:** A set of questions and solutions for Year 9 students focused on 'Pythagoras' Theorem in 3D' under the "Measurement and Geometry" strand, tailored to the Australian Curriculum:

## 1. Understanding Pythagoras in 3D:

**a) Explain how the Pythagorean theorem extends to three dimensions.**

**Solution:**
In 3D, the Pythagorean theorem is used to find the length of the space diagonal (or any diagonal in a rectangular prism) by considering it as the hypotenuse of a right triangle where one side is the length of a diagonal on one of the faces. If you have a rectangular prism with sides a, b, and c, the diagonal d can be calculated with the formula:

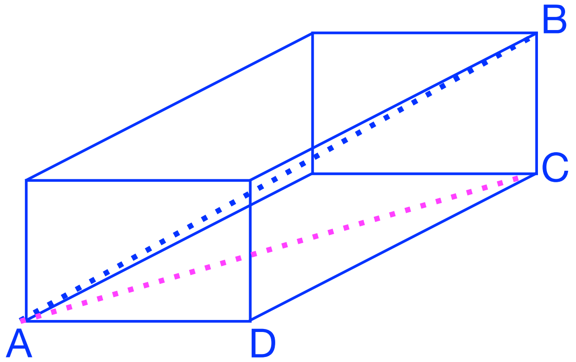
$$d^2 = a^2 + b^2 + c^2,$$

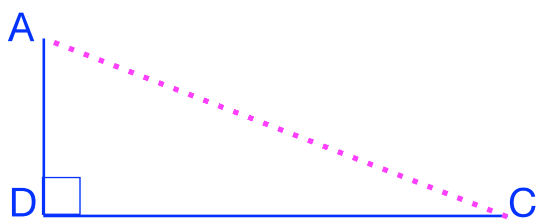
$$\rightarrow d = \sqrt{a^2 + b^2 + c^2}, \equiv d = \sqrt{x^2 + y^2 + z^2}.$$

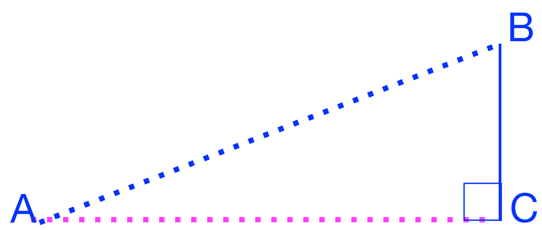
**b) How can you visualise this in a rectangular prism?**

**Solution:**
Imagine dropping a perpendicular from one vertex of the space diagonal A, to the opposite face D, creating a right triangle on that face (ACD), then drop another perpendicular from this point to form another right triangle in 3D (BAC). Thus giving, $(AB)^2 = (AD)^2+(DC)^2+(CB)^2$ .

$$AB = \sqrt{(AD)^2 + (DC)^2 + (CB)^2}.$$



## 2. Finding Diagonals in a Cube:

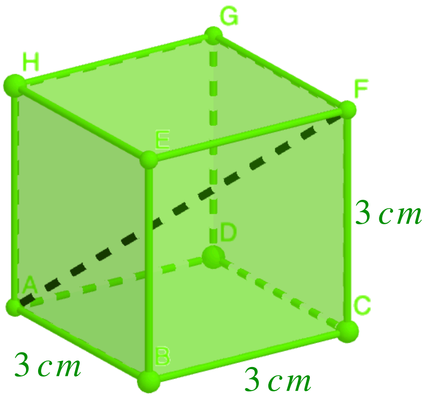
**a) Calculate the length of the space diagonal in a cube with side length $3\,cm$ .**

**Solution:**
For a cube, where all sides are equal $(a = b = c)$ ,

The space diagonal $d$ is :

$$d = \sqrt{a^2 + b^2 + c^2}$$

$$d = \sqrt{3^2 + 3^2 + 3^2}$$
$$= \sqrt{3 \times (3^2)}$$
$$= \sqrt{3^2 \times 3}$$
$$= \cancel{\sqrt{3^2}} \times \sqrt{3}$$
$$= 3 \times \sqrt{3}$$
$$= 3\sqrt{3}$$
$$\approx 5.20\,cm\,.$$

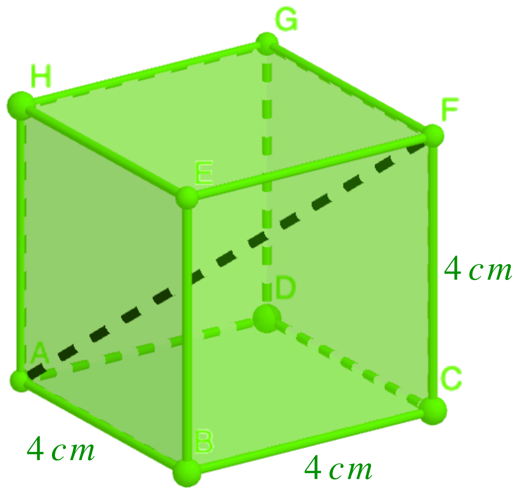



**b) Calculate the length of the space diagonal in a cube with side length $4\,cm$ .**

**Solution:**
The space diagonal $d$ is :

$$d = \sqrt{a^2 + b^2 + c^2}$$

$$d = \sqrt{4^2 + 4^2 + 4^2}$$
$$= \sqrt{3 \times (4^2)}$$
$$= \sqrt{3 \times 4^2}$$
$$= \sqrt{3} \times \sqrt{4^2}$$
$$= \sqrt{3} \times 4$$
$$= 4 \times \sqrt{3}$$
$$= 4\sqrt{3}$$
$$\approx 6.93\,cm\,.$$



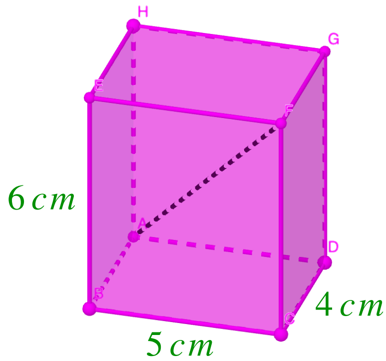
## 3. Diagonals in Rectangular Prisms:

**Find the length of the diagonal of a rectangular prism with dimensions $4\,cm, 5\,cm,$ and $6\,cm$.**

**Solution:**
Using the 3D Pythagorean theorem :

$$d = \sqrt{a^2 + b^2 + c^2}$$

$$\begin{aligned} d &= \sqrt{4^2 + 5^2 + 6^2} \\ &= \sqrt{16 + 25 + 36} \\ &= \sqrt{77} \\ &\approx 8.77\,cm\,. \end{aligned}$$



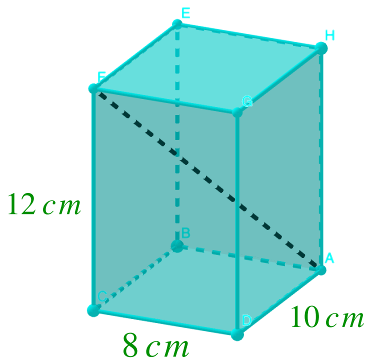

## 4. Practical Application:

**A miniature room has dimensions $8\,cm$ by $10\,cm$ by $12\,cm$. You want to hang a zip-line from one corner on the floor across the room to the opposite corner on the ceiling, how long in the line?**

**Solution:**
This is the space diagonal:

$$d = \sqrt{a^2 + b^2 + c^2}$$

$$\begin{aligned} d &= \sqrt{8^2 + 10^2 + 12^2} \\ &= \sqrt{64 + 100 + 144} \\ &= \sqrt{308} \\ &\approx 17.55\,cm\,. \end{aligned}$$



## 5. Combining 2D Pythagoras and 3D space:

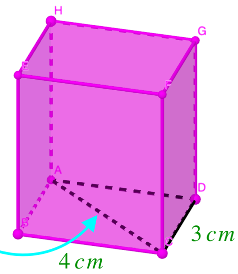
**A rectangular prism has a base with dimensions of $3\,cm$ by $4\,cm$, and a height of $5\,cm$. Find the length of the longest diagonal.**

**Solution:**

$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2 + b^2}$$

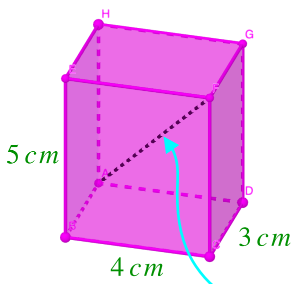
First, find the diagonal of the base (2D Pythagorean):

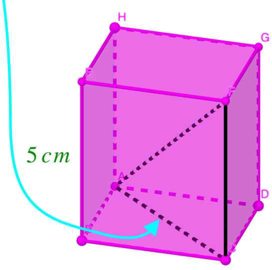
$$\begin{aligned} d_{base} &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5\,cm\,. \end{aligned}$$




Now use this as one side for the 3D calculation:

$$\begin{aligned} d_{space} &= \sqrt{5^2 + 5^2} \\ &= \sqrt{25 + 25} \\ &= \sqrt{50} \\ &= \sqrt{25 \times 2} \\ &= \sqrt{25} \times \sqrt{2} \\ &= 5\sqrt{2} \\ &\approx 7.07\,cm\,. \end{aligned}$$



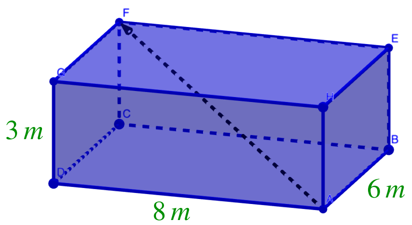

## 6. Real-World Scenario:

**Calculate the length of a wire needed to reach from one bottom corner of a rectangular room to the diagonally opposite top corner, if the room's dimensions are $6\,m$, by $8\,m$, by $3\,m$ high?**

**Solution:**

$$d = \sqrt{a^2 + b^2 + c^2}$$

This scenario requires us to use the 3D Pythagorean theorem:




$$\begin{aligned} d &= \sqrt{6^2 + 8^2 + 3^2} \\ &= \sqrt{36 + 64 + 9} \\ &= \sqrt{109} \\ &\approx 10.44\,metres\,. \end{aligned}$$

## 7. Problem-Solving with Multiple Steps:

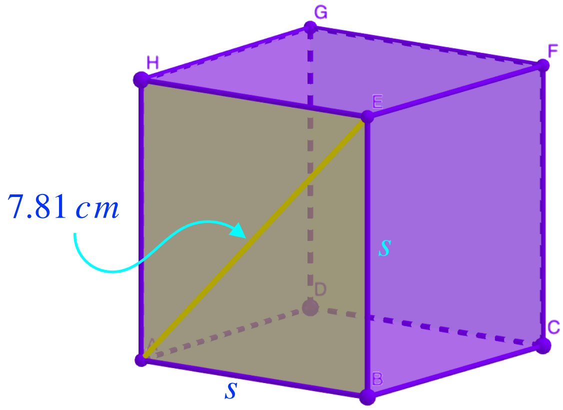
**A designer needs to know the length of a diagonal of a cube for a 3D art installation. If one face of the cube has a diagonal of $7.81\ cm$, what is the length of the space diagonal?**

**Solution:**

$$c^2 = a^2 + b^2$$
$$c = 7.81\ cm\ .$$
$$d^2 = x^2 + y^2 + z^2$$

First, find the side length of the cube using 2D Pythagoras:

$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2 + b^2}$$
$$c = \sqrt{s^2 + s^2}$$
$$c = \sqrt{2s^2}$$




$$7.81 = \sqrt{2s^2}$$
$$(7.81)^2 = (\cancel{\sqrt{2s^2}})^{\cancel{2}}$$
$$7.81^2 = 2s^2$$
$$2s^2 = 7.81^2$$
$$\frac{\cancel{2}s^2}{\cancel{2}} = \frac{7.81^2}{2}$$
$$s^2 = \frac{7.81^2}{2}$$
$$\cancel{\sqrt{s^2}} = \sqrt{\frac{7.81^2}{2}}$$
$$= \sqrt{\frac{7.81^2}{2}}$$
$$s \approx 5.5\ cm\ .$$

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Now calculate the space diagonal:

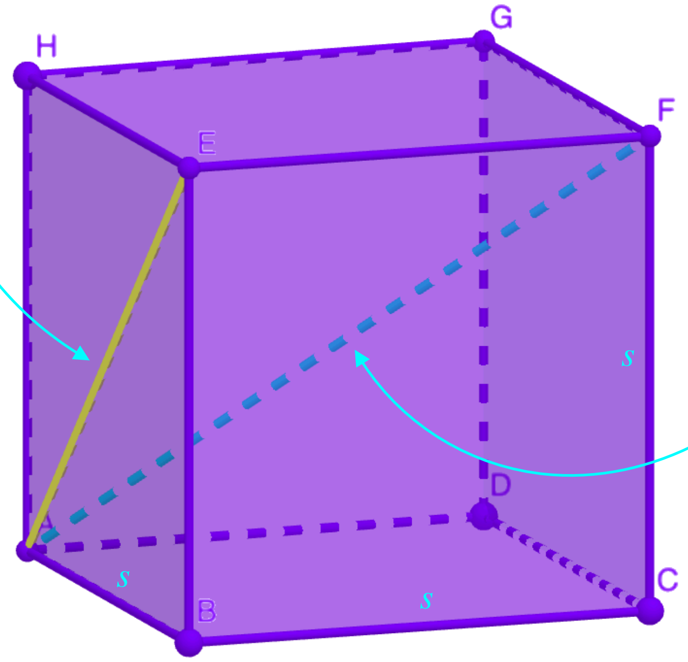
$$d^2 = x^2 + y^2 + z^2$$

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$s \equiv x \equiv y \equiv z \approx 5.5\,cm\,.$$

$$\begin{aligned} d &= \sqrt{5.5^2 + 5.5^2 + 5.5^2} \\ &= \sqrt{3 \times 5.5^2} \\ &= \sqrt{3} \times \sqrt{5.5^2} \\ &= \sqrt{3} \times 5.5 \\ &= 5.5 \times \sqrt{3} \\ &= 5.5\sqrt{3} \\ &\approx 9.53\,cm\,. \end{aligned}$$

$7.81\,cm$




## 8. Pythagorean Triplets in 3D (Quadruplets):

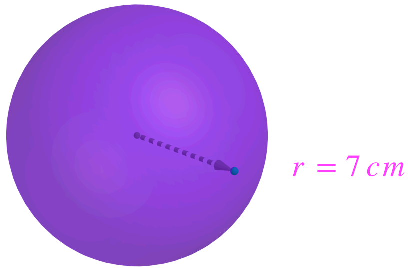
**Examples:** $(1,\ 2,\ 2,\ 3)$ , $(2,\ 3,\ 6,\ 7)$ , $(3,\ 4,\ 12,\ 13)$ . **E.g.** $a^2 + b^2 + c^2 = d^2\ :\ 1^2 + 2^2 + 2^2 = 3^2$ **.**

## 9. Spheres

**a) Find the volume of a sphere with radius $7\,cm$.**

**Solution:**

$$\begin{aligned} V_{sphere} &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times 3.14 \times 7^3 \\ &= \frac{4}{3} \times 3.14 \times 343 \\ &\approx 1{,}436\,cm^3. \end{aligned}$$

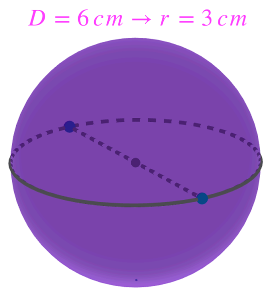



**b) Find the surface area of a sphere with a diameter of $6\,cm$. Use $\pi \approx 3.14$.**

**Solution:**

$$\begin{aligned} \text{Radius, } r &= \frac{6}{2} \\ &= 3cm, \end{aligned}$$

$$\begin{aligned} A_{sphere} &= 4\pi r^2 \\ &= 4 \times 3.14 \times 3^2 \\ &= 4 \times 3.14 \times 9 \\ &= 113.04\,cm^2. \end{aligned}$$




## 10. Cones

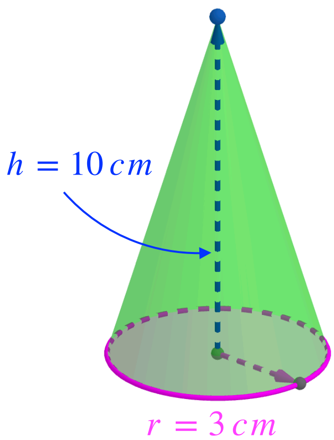
**a) Calculate the volume of a cone with a radius of $3\,cm$ and a height of $10\,cm$. Use $\pi \approx 3.14$.**

**Solution:**

$$V_{cone} = \frac{1}{3}\pi r^2 h$$

$$r = 3\,cm.$$
$$h = 10\,cm.$$

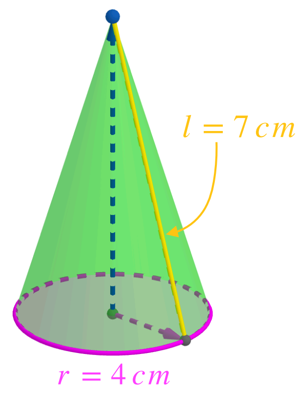
$$\begin{aligned} &= \frac{1}{3} \times 3.14 \times 3^2 \times 10 \\ &= \frac{1}{3} \times 3.14 \times 9 \times 10 \\ &= 94.2\,cm^3. \end{aligned}$$




**b) A cone has a base radius of $4\,cm$ and a slant height of $5\,cm$. Calculate the surface area of the cone (excluding the base). Use $\pi \approx 3.14$.**

**Solution:**

$$SA_{cone} = \pi r l$$
$$r = 4\,cm\,.$$
$$l = 5\,cm\,.$$




The surface area of the cone (excluding the base) involves only the lateral surface area:

$$[\text{Lateral Surface Area}] = \pi r l$$
$$= 3.14 \times 4 \times 5$$
$$= 62.8\,cm^2\,.$$

## 11. Understanding Geometric Proofs:

**a) What is a geometric proof?**

**Solution:**
A geometric proof is a logical argument that uses previously established facts, definitions, postulates, and theorems to show that a particular statement is true in geometry.

**b) Why are geometric proofs important in mathematics?**

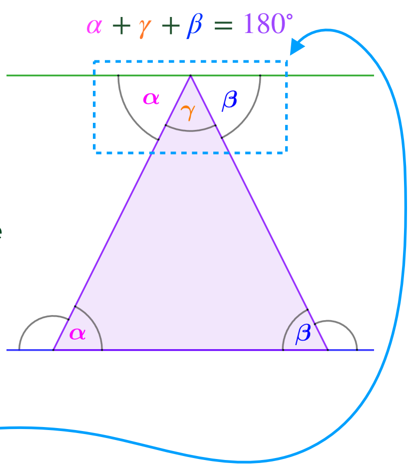
**Solution:**
Proofs are crucial because they provide a systematic way to verify the truth of geometric statements, ensuring the accuracy and consistency of mathematical knowledge. They develop logical reasoning and critical thinking skills.

## 12. Basic Proofs Involving Triangles:

**a) Prove, that the sum of the interior angles in a triangle, add up to** $180\,degrees$**.**

**Solution:**

Draw a triangle and extend one side to create a straight line (blue line at bottom). Draw a line parallel to the one side through the opposite vertex (green line at top).




This forms two angles $\alpha$ and $\beta$ at top, with the extended side (green line), that are equal to the angles at the vertices of the triangle due to alternate interior angles ( $\alpha$ and $\beta$ at bottom, inside triangle ).

The straight line is $180\,degrees$, and the two angles on the line outside the triangle plus the third angle inside the triangle add up to $180^\circ$, hence the sum of the triangle's angles equal $180^\circ$. I.e. $\alpha + \gamma + \beta = 180^\circ$.

**b) Prove the Isosceles Triangle Theorem: If two sides of a triangle are equal, then the angles opposite these sides are equal.**

**Solution:**
Draw an isosceles triangle with equal sides $AB = AC$.

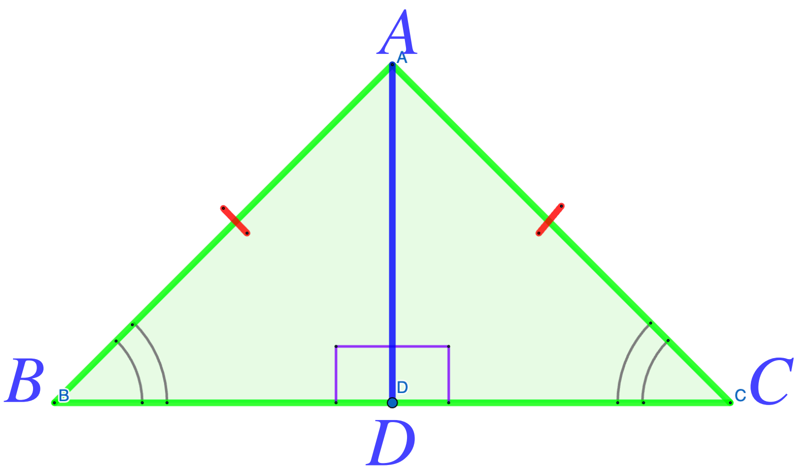
Draw the altitude from vertex $A$ to side $BC$, meeting at $D$ (blue line). This altitude bisects $BC$ into two equal segments $BD = DC$ because it's perpendicular to $BC$.

Now, $\triangle ABD$ and $\triangle ACD$ share side $AD$, have $AB = AC$, and $BD = DC$.

By SSS (Side-Side-Side) congruence, $\triangle ABD \cong \triangle ACD$, (Triangle ABD is congruent to triangle ACD).

*This symbol ( $\cong$ ) means congruent.*

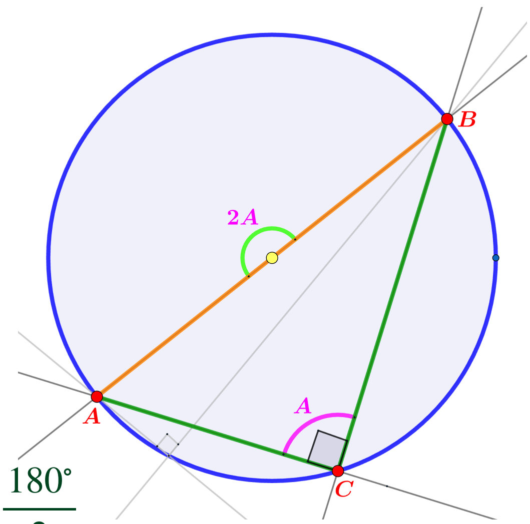
Therefore, $\angle ABD = \angle ACD$, proving the theorem.




## 13. Complex Proofs, Circle Theorems:

**a) Prove the Angle in a Semicircle Theorem: An angle inscribed in a semicircle is a right angle (Thale's Theorem).**

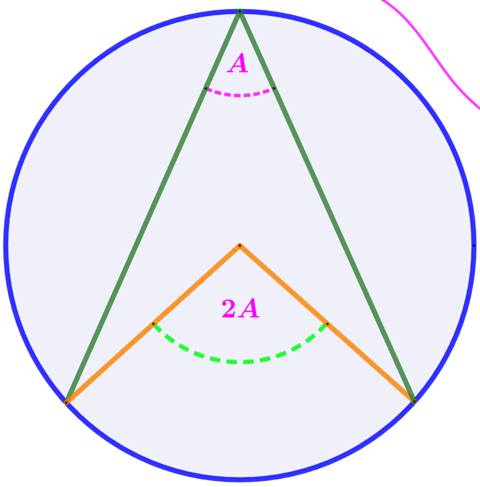
**Solution:**
Consider a semicircle with diameter $AB$, and an inscribed angle $\angle ACB$ (pink line). Draw the line from $C$ to $A$ and $C$ to $B$ (dark green lines).

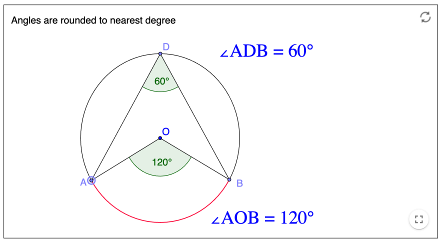


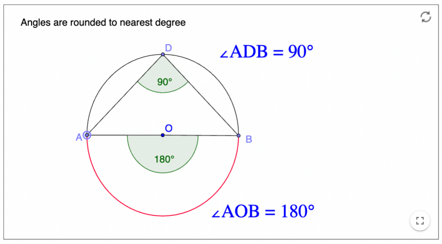

Since $AB$ (orange line) is the diameter, $\angle ACB$ (magenta angle) subtends the arc at $AB$ (orange line), i.e. $\angle AOB$ (bright green angle), with $O$ being the circle centre . The angle at the centre (bright green line) subtended by this arc is $180^\circ$ (since it's a semicircle). The angle at the circumference is half the angle at the centre for the same arc, i.e a right-angle (reverse of double angle theorem, [see diagram below]), so:

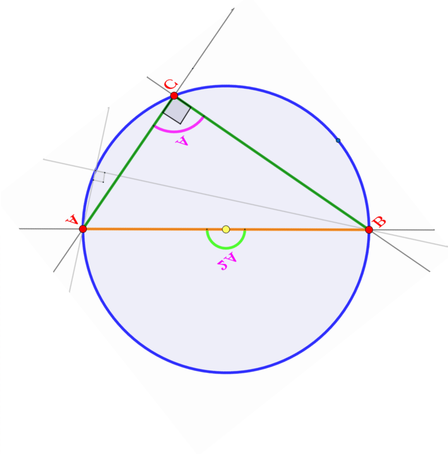
$$\angle ACB = \frac{180^\circ}{2} = 90^\circ.$$

**GeoGebra Interactive,** *Available 13 March 2025, from* **: [https://www.geogebra.org/m/ekqyj9dx](https://www.geogebra.org/m/ekqyj9dx)**



**b) Prove that the opposite angles of a cyclic quadrilateral sum to** $180\,degrees$**.** *(Tricky!)*

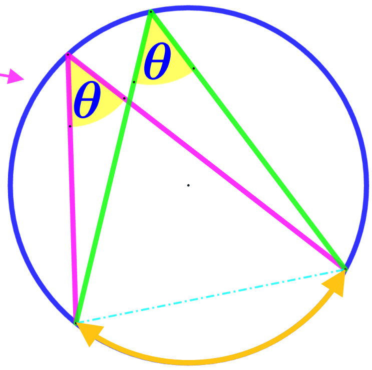
**Solution:**

Draw a cyclic quadrilateral $ABCD$. Draw the diagonals $AC$ (blue line) and $BD$ (yellow line) intersecting at $O$. [The angles ($\theta$) subtended by the same arc at the circumference are equal,] so:

We know :

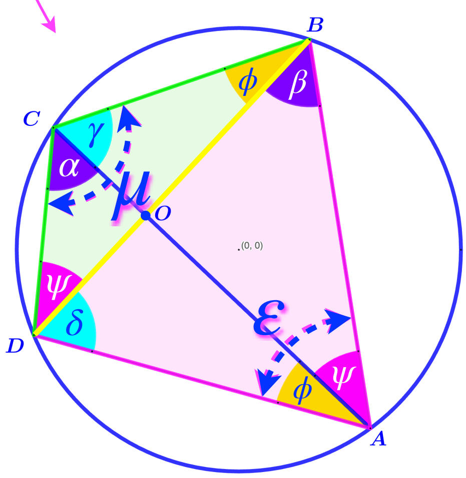
$$\alpha = \beta,$$
$$\gamma = \delta.$$
$$\mu = \alpha + \gamma,$$
$$\varepsilon = \phi + \psi.$$

$$\rightarrow \varepsilon - \psi = \phi + \cancel{\psi} - \cancel{\psi}$$
$$\varepsilon - \psi = \phi$$
$$\phi = \varepsilon - \psi$$




Also :

The angles at the yellow sectors are equal, ($\phi$) &
the angles at the magenta sectors are equal ($\psi$).




The pink triangle ($DAB$) shows:

$$\beta + \delta + \varepsilon = 180^\circ \ (Angles\ in\ \triangle = 180^\circ)$$

$$\beta + \delta + \varepsilon = 180^\circ$$
$$\rightarrow \alpha + \gamma + \varepsilon = 180^\circ$$
$$\alpha + \gamma + \cancel{\varepsilon} - \cancel{\varepsilon} = 180^\circ - \varepsilon$$
$$\alpha + \gamma = 180^\circ - \epsilon$$

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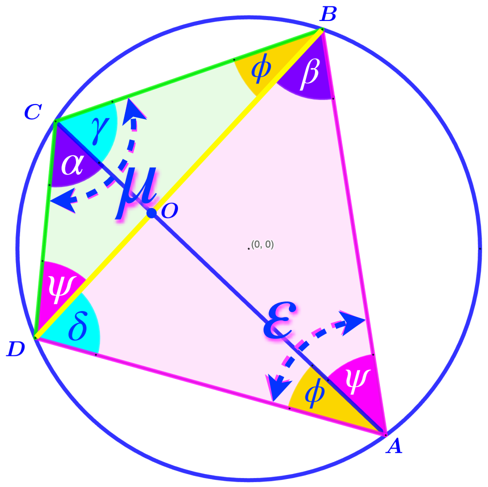
The green triangle ( $DCB$ ) shows:

$$\mu + \psi + \phi = 180^\circ \ (Angles\ in\ \triangle = 180^\circ)$$




$$\mu + \psi + \phi = 180^\circ$$
$$\cancel{\mu} - \cancel{\mu} + \psi + \phi = 180^\circ - \mu$$
$$\phi + \psi = 180^\circ - \mu$$

Now, $\mu = \alpha + \gamma$,
and, $\varepsilon = \phi + \psi$.

$$\rightarrow \mu + \varepsilon = (\alpha + \gamma) + (\phi + \psi)$$
$$\mu + \varepsilon = (180^\circ - \varepsilon) + (180^\circ - \mu)$$
$$\mu + \varepsilon = 360^\circ - \varepsilon - \mu$$
$$\mu + \varepsilon + \varepsilon + \mu = 360^\circ - \cancel{\varepsilon} - \cancel{\mu} + \cancel{\varepsilon} + \cancel{\mu}$$
$$2\mu + 2\varepsilon = 360^\circ$$
$$\frac{\cancel{2}(\mu + \varepsilon)}{\cancel{2}} = \frac{360^\circ}{2}$$

$$\therefore \mu + \varepsilon = 180^\circ.$$

**OR**

Visually, $\varepsilon + \delta + \beta = 180^\circ$, ( Pink triangle ) and,
$\mu + \psi + \phi = 180^\circ$, ( Green triangle ),

So the Pink triangle $+$ the Green triangle $= 360^\circ$, and
$360^\circ -$ ( Angle at $D$ $+$ Angle at $B$ ) $= 180^\circ$,
$\rightarrow$ Angle at $D$ $+$ Angle at $B$ $= 180^\circ$.

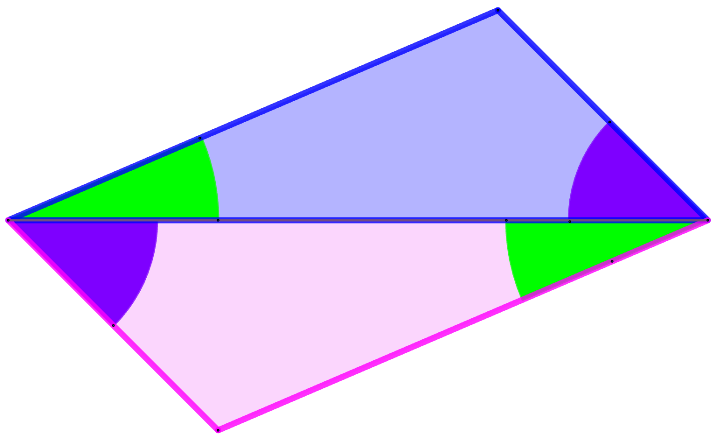
$\therefore$ Pink triangle $+$ Green triangle $-$ ( Angle at $D$ $+$ Angle at $B$ ) $= 360^\circ - 180^\circ$

$$(\varepsilon + \delta + \beta) + (\mu + \psi + \phi) - (\psi + \delta) - (\phi + \beta) = 180^\circ$$
$$\varepsilon + \delta + \beta + \mu + \psi + \phi - \psi - \delta - \phi - \beta = 180^\circ$$
$$\varepsilon + \cancel{\delta} + \cancel{\beta} + \mu + \cancel{\psi} + \cancel{\phi} - \cancel{\psi} - \cancel{\delta} - \cancel{\phi} - \cancel{\beta} = 180^\circ$$
$$\varepsilon + \mu = 180^\circ.$$

**GeoGebra Interactive**, *Available 13 March 2025, from* : https://www.geogebra.org/m/Npdh7EJ4

## 14. Proofs Involving Congruence:

**Prove that if two angles and the included side of one triangle (blue triangle) are equal to two angles and the included side of another triangle (red triangle), then the triangles are congruent by the ASA criterion.**



**Solution:**

Visually, it can be seen that the blue triangle and the red triangle are the same, except one has been rotated $180°$.

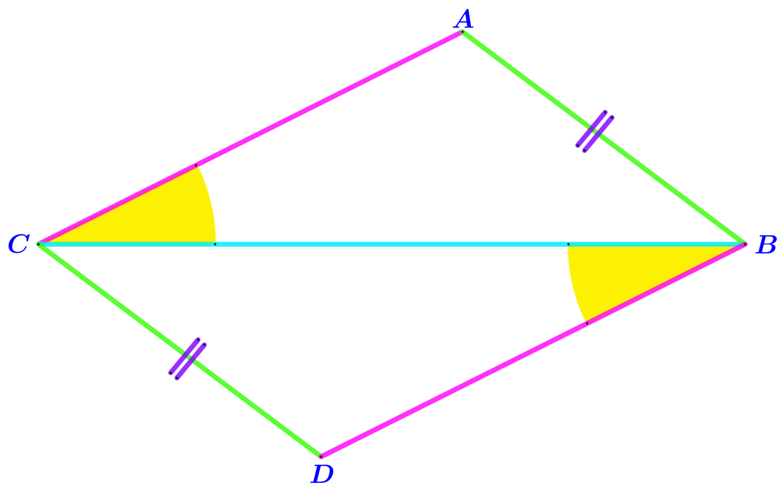
Therefore, the red triangle and the blue triangle must be congruent, (as they are the same but one is rotated compared to the other), by the ASA criterion.

**Solution:**

Given triangles $CAB$ and $CDB$ where :

$$AB \parallel CD,$$
$$\rightarrow \angle ACB \cong \angle DCB \text{ ( Alternate interior angles )}$$
$$\text{And, } CB \cong BC.$$

$$\therefore \triangle ABC \cong \triangle DCB.$$




**OR**

Given triangles $CAB$ and $CDB$ where :

$\angle A = \angle D$,
$\angle B = \angle E$, and $AB = DC$.

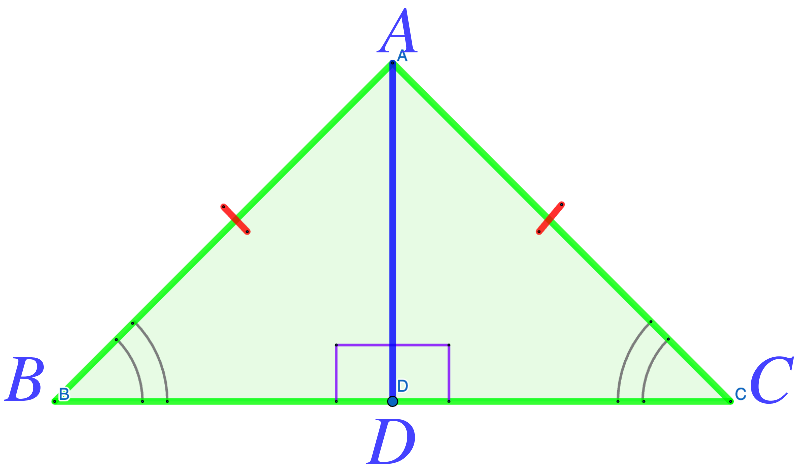
Since the included side $AB$ (or $DC$) is common,
we have all necessary conditions for $ASA$ (Angle-Side-Angle) congruence.

Therefore, by the $ASA$ congruence criterion,
$\triangle ABC \cong \triangle BCD$.

## 15. Practical Application:

**In designing a garden, how would you use geometric proofs to ensure the symmetry of a triangular flower bed where two sides are equal?**

**Solution:**
By proving that if two sides of the triangle are equal, the angles opposite these sides are also equal (Isosceles Triangle Theorem), you ensure symmetry. This can be demonstrated in the garden by ensuring that the angles at the base of the triangular bed are equal, which will visually balance the design. *(See Question 2b. for Isosceles Triangle Theorem).*



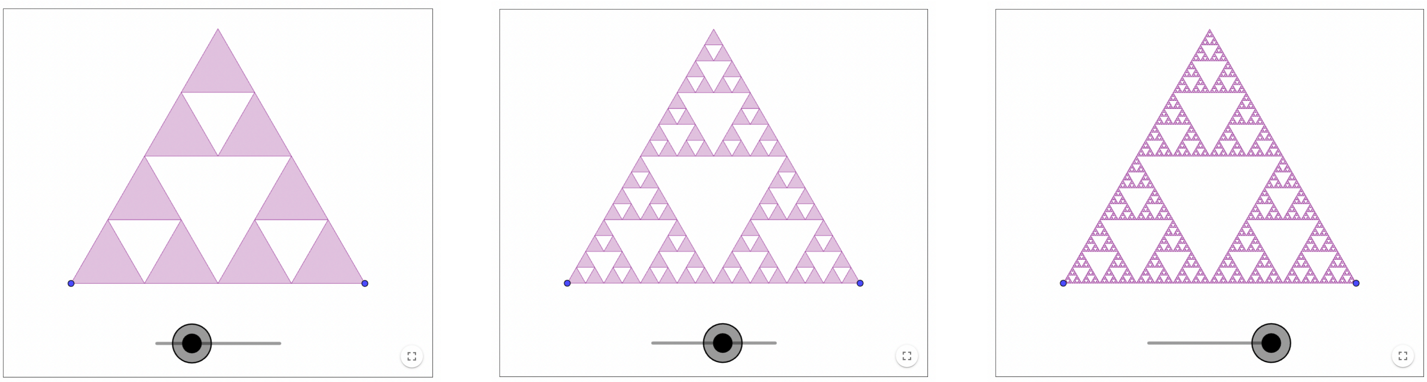

## 16. Logical Reasoning in Proofs:

**Explain the importance of logical steps in geometric proofs.**

**Solution:**
Each step in a proof must logically follow from the previous one, using only accepted axioms or previously proven theorems. This logical progression ensures that the conclusion is undeniable, teaching students to think critically and methodically.

## 17. Do you know why?

Why do most exercise books, pads, and graph paper, have light blue lines on them? **Solution:** Facsimiles.:o



**Sierpinski triangle (Sierpinski gasket) - Introduction to Chaos and Fractals.**

**GeoGebra Interactive,** *Available 13 March 2025, from* **:** https://www.geogebra.org/m/hh9mnbcr

## Additional Notes for Teachers:

**Learning Outcomes:** Students should understand the properties of circles, calculate various aspects like circumference, area, arc length, and sector area, and apply circle theorems. Students should be able to construct logical geometric proofs, understand the principles behind various geometric theorems, and apply these in problem-solving scenarios.
**Teaching Strategies:** Use physical circles or digital tools to illustrate concepts like radius, circumference, and angles.
Engage students with activities where they measure circles or use compasses to draw them.
Discuss and prove circle theorems to deepen understanding of geometric properties.
Encourage students to write out each step of their proof, explaining why each statement follows from the last.
Use visual aids or dynamic geometry software to illustrate the steps of proofs.
Discuss historical proofs or have students attempt to prove simple theorems independently.
**Assessment:** Assess through problems that require calculation, application of theorems, and real-world problems involving circles. Evaluate through tasks where students must construct proofs, identify flaws in given proofs, or apply geometric theorems in practical contexts.
**Resources:** Compasses, protractors, string for measuring circumference, or geometry software for dynamic exploration. Geometry sets for drawing, whiteboards for collaborative proof construction, or software for interactive geometry exploration.

This question set aligns with the Australian Curriculum for Year 9, focusing on the proficiencies of understanding, fluency, problem-solving, and reasoning in measurement and geometry, specifically in the context of circles, and geometric proofs.